

Homology of torus spaces with acyclic proper faces of the orbit space

Anton Ayzenberg

ABSTRACT. Let X be $2n$ -dimensional compact manifold and $T^n \curvearrowright X$ be a locally standard action of a compact torus. The orbit space X/T is a manifold with corners. Suppose that all proper faces of X/T are acyclic. In the paper we study the homological spectral sequence ${}^X E_{*,*}^* \Rightarrow H_*(X)$ corresponding to the filtration of X by orbit types. When the free part of the action is not twisted, we describe the whole spectral sequence ${}^X E_{*,*}^*$ in terms of homology and combinatorial structure of the orbit space X/T . In this case we describe the kernel and the cokernel of the natural map $\mathbb{k}[X/T]/(l.s.o.p.) \rightarrow H^*(X)$, where $\mathbb{k}[X/T]$ is a face ring of X/T and $(l.s.o.p.)$ is the ideal generated by a linear system of parameters (this ideal appears as the image of $H^{>0}(BT)$ in $H_T^*(X)$). There exists a natural double grading on $H_*(X)$, which satisfies bigraded Poincaré duality. This general theory is applied to compute homology groups of origami toric manifolds with acyclic proper faces of the orbit space. A number of natural generalizations is considered. These include Buchsbaum simplicial complexes and posets. h' - and h'' -numbers of simplicial posets appear as the ranks of certain terms in the spectral sequence ${}^X E_{*,*}^*$. In particular, using topological argument we show that Buchsbaum posets have nonnegative h'' -vectors. The proofs of this paper rely on the theory of cellular sheaves. We associate to a torus space certain sheaves and cosheaves on the underlying simplicial poset, and observe an interesting duality between these objects. This duality seems to be a version of Poincaré–Verdier duality between cellular sheaves and cosheaves.

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1. Introduction

Let M be $2n$ -dimensional compact manifold with a locally standard action of a compact torus T^n . This means, by definition, that the action of T^n on M^{2n} is locally modeled by a standard coordinate-wise action of T^n on \mathbb{C}^n . Since \mathbb{C}^n/T^n can be identified with a nonnegative cone \mathbb{R}_{\geq}^n , the quotient space $Q = M/T$ has a natural structure of a compact manifold with corners. The general problem is the following:

PROBLEM 1. *Describe the (co)homology of M in terms of combinatorics and topology of the orbit space Q and the local data of the action.*

The answer is known in the case when Q and all its faces are acyclic (so called homology polytope) [9]. In this case the equivariant cohomology ring of M coincides with the face ring of simplicial poset S_Q dual to Q , and the ordinary cohomology has description similar to that of toric varieties or quasitoric manifolds:

$$H^*(M; \mathbb{k}) \cong \mathbb{k}[S_Q]/(\theta_1, \dots, \theta_n), \quad \deg \theta_i = 2.$$

In this case $\mathbb{k}[S_Q]$ is Cohen–Macaulay and $\theta_1, \dots, \theta_n$ is a linear regular sequence determined by the characteristic map on Q . In particular, cohomology vanishes in odd degree, and $\dim H^{2i}(M) = h_i(S_Q)$.

In general, there is a topological model of a manifold M , called canonical model. The construction is the following. Start with a nice manifold with corners Q , consider a principal T^n -bundle Y over Q , and then consider the quotient space $X = Y/\sim$ determined by a characteristic map [17, def. 4.2]. It is known that X is a manifold with locally standard torus action, and every manifold with l.s.t.a. is equivariantly homeomorphic to such canonical model. Thus it is sufficient to work with canonical models to answer Problem 1.

In this paper we study the case when all proper faces of Q are acyclic, but Q itself may be arbitrary. Homology of X can be described by the spectral sequence ${}^X E_{*,*}^r$ associated to the filtration of X by orbit types:

$$(1.1) \quad X_0 \subset X_1 \subset \dots \subset X_{n-1} \subset X_n = X, \quad \dim X_i = 2i.$$

This filtration is covered by the filtration of Y :

$$(1.2) \quad Y_0 \subset Y_1 \subset \dots \subset Y_{n-1} \subset Y_n = Y, \quad X_i = Y_i/\sim.$$

We prove that most entries of the second page ${}^XE_{*,*}^2$ coincide with corresponding entries of ${}^YE_{*,*}^2$ (Theorem 1). When Y is a trivial T^n -bundle, $Y = Q \times T^n$, this observation allows to describe ${}^XE_{*,*}^*$ completely in terms of topology and combinatorics of Q (Theorem 2, statement 5.3 and Theorem 3). This answers Problem 1 additively. From this description in particular follows that Betti numbers of X do not depend on the choice of characteristic map. We hope, that this technic will lead to the description of cohomology multiplication in $H^*(X)$ as well.

Another motivation for this paper comes from a theory of Buchsbaum simplicial complexes and posets. The notions of h' - and h'' -vectors of simplicial poset S first appeared in combinatorial commutative algebra [15, 13]. These invariants emerge naturally in the description of homology of X (Theorems 3 and 4). The space $X = Y/\sim$ can be constructed not only in the case when Q is a manifold with corners (“manifold case”), but also in the case when Q is a cone over geometric realization of simplicial poset S (“cone case”). In the cone case, surely, X may not be a manifold. But there still exists filtration (1.1), and homology groups of X can be calculated by the same method as for manifolds, when S is Buchsbaum. In the cone case we prove that $\dim {}^XE_{i,i}^\infty = h''(S)$ (Theorem 5). Thus, in particular, h'' -vector of Buchsbaum simplicial poset is nonnegative. This result is proved in commutative algebra by completely different methods [13].

The exposition of the paper is built in such way that both situations: manifolds with acyclic faces, and cones over Buchsbaum posets are treated in a common context. In order to do this we introduce the notion of Buchsbaum pseudo-cell complex which is very natural and includes both motivating examples. A theory of cellular sheaves over simplicial posets is used to prove basic theorems. The coincidence of most parts of ${}^XE_{*,*}^r$ and ${}^YE_{*,*}^r$ follows from the Key lemma (lemma 5.1) which is an instance of general duality between certain sheaves and cosheaves (Theorem 6). In the manifold case this duality can be deduced from Verdier duality for cellular sheaves, described in [6].

The paper is organized as follows. Section 2 contains preliminaries on simplicial posets and cellular sheaves. In section 3 we introduce the notion of simple pseudo-cell complex and describe spectral sequences associated to filtrations by pseudo-cell skeleta. Section 4 is devoted to torus spaces over pseudo-cell complexes. The main results (Theorems 1–5) are stated in section 5. The rest of section 5 contains the description of homology of X . There is an additional grading on homology groups, and in the manifold case there is a bigraded Poincare duality. Section 6 contains a sheaf-theoretic discussion of the subject. In this section we prove Theorem 6 which can be considered as a version of cellular Verdier duality. This proves the Key lemma, from which follow Theorems 1 and 2. Section 7 is devoted to the combinatorics of simplicial posets. In this section we recall combinatorial definitions of f -, h -, h' - and h'' -vectors and prove Theorems 3–5. The structure of equivariant cycles and cocycles of a manifold X with locally standard torus action is the subject of section

8. There exists a natural map $\mathbb{k}[S]/(\theta_1, \dots, \theta_n) \rightarrow H^*(X)$, where $(\theta_1, \dots, \theta_n)$ is a linear system of parameters, associated to a characteristic map. In general (i.e. when Q is not a homology polytope) this map may be neither injective nor surjective. The kernel of this map is described by corollary 8.8. The calculations for some particular examples are gathered in section 9. The main family of nontrivial examples is the family of origami toric manifolds with acyclic proper faces of the orbit space.

2. Preliminary constructions

2.1. Preliminaries on simplicial posets. First, recall several standard definitions. A partially ordered set (poset in the following) is called simplicial if it has a minimal element $\emptyset \in S$, and for any $I \in S$, the lower order ideal $S_{\leq I} = \{J \mid J \leq I\}$ is isomorphic to the boolean lattice $2^{[k]}$ (the poset of faces of a simplex). The number k is called the rank of $I \in S$ and is denoted $|I|$. Also set $\dim I = |I| - 1$. A vertex is a simplex of rank 1 (i.e. the atom of the poset); the set of all vertices is denoted by $\text{Vert}(S)$. A subset $L \subset S$, for which $I < J$, $J \in L$ implies $I \in L$ is called a simplicial subposet.

The notation $I <_i J$ is used whenever $I < J$ and $|J| - |I| = i$. If S is a simplicial poset, then for each $I <_2 J \in S_Q$ there exist exactly two intermediate simplices J', J'' :

$$(2.1) \quad I <_1 J', J'' <_1 J.$$

For simplicial poset S a “sign convention” can be chosen. It means that we can associate an incidence number $[J : I] = \pm 1$ to any $I <_1 J \in S$ in such way that for (2.1) holds

$$(2.2) \quad [J : J'] \cdot [J' : I] + [J : J''] \cdot [J'' : I] = 0.$$

The choice of a sign convention is equivalent to the choice of orientations of all simplices.

For $I \in S$ consider the link:

$$\text{lk}_S I = \{J \in S \mid J \geq I\}.$$

It is a simplicial poset with minimal element I . On the other hand, $\text{lk}_S I$ can also be considered as a subset of S . It can be seen that $S \setminus \text{lk}_S I$ is a simplicial subposet. Note, that $\text{lk}_S \emptyset = S$.

Let S' be the barycentric subdivision of S . By definition, S' is a simplicial complex on the set $S \setminus \emptyset$ whose simplices are the chains of elements of S . By definition, the geometric realization of S is the geometric realization of its barycentric subdivision $|S| \stackrel{\text{def}}{=} |S'|$. One can also think about $|S|$ as a CW-complex with simplicial cells [2]. A poset S is called pure if all its maximal elements have equal dimensions. A poset S is pure whenever S' is pure.

DEFINITION 2.1. *Simplicial complex K of dimension $n - 1$ is called Buchsbaum if $\tilde{H}_i(\mathrm{lk}_K I) = 0$ for all $\emptyset \neq I \in K$ and $i \neq n - 1 - |I|$. If K is Buchsbaum and, moreover, $\tilde{H}_i(K) = 0$ for $i \neq n - 1$ then K is called Cohen–Macaulay. Simplicial poset S is called Buchsbaum (Cohen–Macaulay) if S' is a Buchsbaum (resp. Cohen–Macaulay) simplicial complex.*

REMARK 2.2. Whenever the coefficient ring in the notation of (co)homology is omitted it is supposed to be the ground ring \mathbb{k} , which is either a field or the ring of integers.

REMARK 2.3. By [13, Sec.6], S is Buchsbaum whenever $\tilde{H}_i(\mathrm{lk}_S I) = 0$ for all $\emptyset \neq I \in S$ and $i \neq n - 1 - |I|$. Similarly, S is Cohen–Macaulay if $\tilde{H}_i(\mathrm{lk}_S I) = 0$ for all $I \in S$ and $i \neq n - 1 - |I|$. A poset S is Buchsbaum whenever all its proper links are Cohen–Macaulay.

One easily checks that Buchsbaum property implies purity.

2.2. Cellular sheaves. Let $\mathrm{MOD}_{\mathbb{k}}$ be the category of \mathbb{k} -modules. The notation $\dim V$ is used for the rank of a \mathbb{k} -module V .

Each simplicial poset S defines a small category $\mathrm{CAT}(S)$ whose objects are the elements of S and morphisms — the inequalities $I \leq J$. A cellular sheaf [6] (or a stack [12], or a local coefficient system elsewhere) is a covariant functor $\mathcal{A}: \mathrm{CAT}(S) \rightarrow \mathrm{MOD}_{\mathbb{k}}$. We simply call \mathcal{A} a sheaf on S and hope that this will not lead to a confusion, since different meanings of this word do not appear in the paper. The maps $\mathcal{A}(J_1 \leq J_2)$ are called the restriction maps. The cochain complex $(C^*(S; \mathcal{A}), d)$ is defined as follows:

$$C^*(S; \mathcal{A}) = \bigoplus_{i \geq -1} C^i(S; \mathcal{A}), \quad C^i(S; \mathcal{A}) = \bigoplus_{\dim I = i} \mathcal{A}(I),$$

$$d: C^i(S; \mathcal{A}) \rightarrow C^{i+1}(S; \mathcal{A}), \quad d = \bigoplus_{I <_1 I', \dim I = i} [I' : I] \mathcal{A}(I \leq I').$$

By the standard argument involving sign convention (2.2), $d^2 = 0$, thus $(C^*(S; \mathcal{A}), d)$ is a differential complex. Define the cohomology of \mathcal{A} as the cohomology of this complex:

$$(2.3) \quad H^*(S; \mathcal{A}) \stackrel{\mathrm{def}}{=} H^*(C^*(S; \mathcal{A}), d).$$

REMARK 2.4. Cohomology of \mathcal{A} defined this way coincide with any other meaningful definition of cohomology. E.g. the derived functors of the functor of global sections are isomorphic to (2.3) (see [6] for the vast exposition of this subject).

A sheaf \mathcal{A} on S can be restricted to a simplicial subposet $L \subset S$. The complexes $(C^*(L, \mathcal{A}), d)$ and $(C^*(S; \mathcal{A})/C^*(L; \mathcal{A}), d)$ are defined in a usual manner. The latter complex gives rise to a relative version of sheaf cohomology: $H^*(S, L; \mathcal{A})$.

REMARK 2.5. It is standard in topological literature to consider cellular sheaves which do not take values on $\emptyset \in S$, since in general this element has no geometrical meaning. However, this extra value $\mathcal{A}(\emptyset)$ is very important in the considerations of this paper. Thus the cohomology group may be nontrivial in degree $\dim \emptyset = -1$. If a sheaf \mathcal{A} is defined on S , then we often consider its truncated version $\underline{\mathcal{A}}$ which coincides with \mathcal{A} on $S \setminus \{\emptyset\}$ and vanishes on \emptyset .

EXAMPLE 2.6. Let W be a \mathbb{k} -module. By abuse of notation let W denote the (globally) constant sheaf on S . It takes constant value W on $\emptyset \neq I \in S$ and vanishes on \emptyset ; all nontrivial restriction maps are identity isomorphisms. In this case $H^*(S; W) \cong H^*(S) \otimes W$.

EXAMPLE 2.7. A locally constant sheaf valued by $W \in \text{MOD}_{\mathbb{k}}$ is a sheaf \mathcal{W} which satisfies $\mathcal{W}(\emptyset) = 0$, $\mathcal{W}(I) \cong W$ for $I \neq \emptyset$ and all nontrivial restriction maps are isomorphisms.

EXAMPLE 2.8. Let $I \in S$ and $W \in \text{MOD}_{\mathbb{k}}$. Consider the sheaf \mathbb{I}^W defined by

$$(2.4) \quad \mathbb{I}^W(J) = \begin{cases} W, & \text{if } J \geq I \\ 0, & \text{otherwise,} \end{cases}$$

with the restriction maps $\mathbb{I}^W(J_1 \leq J_2)$ either identity on W (when $I \leq J_1$), or 0 (otherwise). Then $\mathbb{I}^W = \mathbb{I}^{\mathbb{k}} \otimes W$ and $H^*(S; \mathbb{I}^W) \cong H^*(S; \mathbb{I}^{\mathbb{k}}) \otimes W$. We have

$$H^*(S; \mathbb{I}^{\mathbb{k}}) \cong H^{*-|I|}(\text{lk}_S I),$$

since corresponding differential complexes coincide.

In the following if \mathcal{A} and \mathcal{B} are two sheaves on S we denote by $\mathcal{A} \otimes \mathcal{B}$ their componentwise tensor product: $(\mathcal{A} \otimes \mathcal{B})(I) = \mathcal{A}(I) \otimes \mathcal{B}(I)$ with restriction maps defined in the obvious way.

EXAMPLE 2.9. As a generalization of the previous example consider the sheaf $\mathbb{I}^{\mathbb{k}} \otimes \mathcal{A}$. Then

$$H^*(S; \mathbb{I}^{\mathbb{k}} \otimes \mathcal{A}) \cong H^{*-|I|}(\text{lk}_S I; \mathcal{A}|_{\text{lk}_S I}).$$

EXAMPLE 2.10. Following [12], define i -th local homology sheaf \mathcal{U}_i on S by setting $\mathcal{U}_i(\emptyset) = 0$ and

$$(2.5) \quad \mathcal{U}_i(J) = H_i(S, S \setminus \text{lk}_S J)$$

for $J \neq \emptyset$. The restriction maps $\mathcal{U}_i(J_1 < J_2)$ are induced by inclusions $\text{lk}_S J_2 \hookrightarrow \text{lk}_S J_1$. A poset S is Buchsbaum if and only if $\mathcal{U}_i = 0$ for $i < n - 1$.

DEFINITION 2.11. *Buchsbaum poset S is called homology manifold (orientable over \mathbb{k}) if its local homology sheaf \mathcal{U}_{n-1} is isomorphic to the constant sheaf \mathbb{k} .*

If $|S|$ is a compact closed orientable topological manifold then S is a homology manifold.

2.3. Cosheaves. A cellular cosheaf (see [6]) is a contravariant functor $\hat{\mathcal{A}}: \text{CAT}^{op}(S) \rightarrow \text{MOD}_{\mathbb{k}}$. The homology of a cosheaf is defined similar to cohomology of sheaves:

$$\begin{aligned} C_*(S; \hat{\mathcal{A}}) &= \bigoplus_{i \geq -1} C_i(S; \hat{\mathcal{A}}) & C_i(S; \hat{\mathcal{A}}) &= \bigoplus_{\dim I = i} \mathcal{A}(I) \\ d: C_i(S; \hat{\mathcal{A}}) &\rightarrow C_{i-1}(S; \hat{\mathcal{A}}), & d &= \bigoplus_{I >_1 I', \dim I = i} [I : I'] \hat{\mathcal{A}}(I \geq I'), \\ H_*(S; \hat{\mathcal{A}}) &\stackrel{\text{def}}{=} H_*(C_*(S; \hat{\mathcal{A}}), d). \end{aligned}$$

EXAMPLE 2.12. Each locally constant sheaf \mathcal{W} on S defines the locally constant cosheaf $\hat{\mathcal{W}}$ by inverting arrows, i.e. $\hat{\mathcal{W}}(I) \cong \mathcal{W}(I)$ and $\hat{\mathcal{W}}(I > J) = (\mathcal{W}(J < I))^{-1}$.

3. Buchsbaum pseudo-cell complexes

3.1. Simple pseudo-cell complexes.

DEFINITION 3.1 (Pseudo-cell complex). A *CW-pair* $(F, \partial F)$ will be called *k-dimensional pseudo-cell*, if F is compact and connected, $\dim F = k$, $\dim \partial F \leq k-1$. A (regular finite) *pseudo-cell complex* Q is a space which is a union of an expanding sequence of subspaces Q_k such that Q_{-1} is empty and Q_k is the pushout obtained from Q_{k-1} by attaching finite number of k -dimensional pseudo-cells $(F, \partial F)$ along injective attaching maps $\partial F \rightarrow Q_{k-1}$. The images of $(F, \partial F)$ in Q will be also called *pseudo-cells* and denoted by the same letters.

REMARK 3.2. In general, situations when $\partial F = \emptyset$ or $Q_0 = \emptyset$ are allowed by this definition. Thus the construction of pseudo-cell complex may actually start not from Q_0 but from higher dimensions.

Let $F^\circ = F \setminus \partial F$ denote open cells. In the following we assume that the boundary of each cell is a union of lower dimensional cells. Thus all pseudo-cells of Q are partially ordered by inclusion. We denote by S_Q the poset of faces with the reversed order, i.e. $F <_{S_Q} G$ iff $G \subseteq \partial F \subset F$. To distinguish abstract elements of poset S_Q from faces of Q the former are denoted by $I, J, \dots \in S_Q$, and corresponding faces — $F_I, F_J, \dots \subseteq Q$.

DEFINITION 3.3. A pseudo-cell complex Q , $\dim Q = n$ is called *simple* if S_Q is a simplicial poset of dimension $n-1$ and $\dim F_I = n-1-\dim I$ for all $I \in S_Q$.

Thus for every face F , the upper interval $\{G \mid G \supseteq F\}$ is isomorphic to a boolean lattice $2^{[\text{codim } F]}$. In particular, there exists a unique maximal pseudo-cell F_\emptyset of dimension n , i.e. Q itself. In case of simple pseudo-cell complexes we adopt the following naming convention: pseudo-cells different from $F_\emptyset = Q$ are called faces, and faces of codimension 1 — facets. Facets correspond to vertices (atoms) of S_Q . Each face F is contained in exactly $\text{codim } F$ facets. In this paper only simple pseudo-cell complexes are considered.

EXAMPLE 3.4. Nice (compact connected) manifolds with corners as defined in [9] are examples of simple pseudo-cell complexes. Each face F is itself a manifold and ∂F is the boundary in a common sense.

EXAMPLE 3.5. Each pure simplicial poset S determines a simple pseudo-cell complex $P(S)$ such that $S_{P(S)} = S$ by the following standard construction. Consider the barycentric subdivision S' and construct the cone $P(S) = |\text{Cone } S'|$. By definition, $\text{Cone } S'$ is a simplicial complex on the set S and k -simplices of $\text{Cone } S'$ have the form $(I_0 < I_1 < \dots < I_k)$, where $I_i \in S$. For each $I \in S$ consider the pseudo-cell:

$$F_I = |\{(I_0 < I_1 < \dots) \in \text{Cone } S' \text{ such that } I_0 \geq I\}| \subset |\text{Cone } S'|$$

$$\partial F_I = |\{(I_0 < I_1 < \dots) \in \text{Cone } S' \text{ such that } I_0 > I\}| \subset |\text{Cone } S'|$$

Since S is pure, $\dim F_I = n - \dim I - 1$. These sets define a pseudo-cell structure on $P(S)$. One shows that $F_I \subset F_J$ whenever $J < I$. Thus $S_{P(S)} = S$. Face F_I is called dual to $I \in S$. The filtration by pseudo-cell skeleta

$$(3.1) \quad \emptyset = Q_{-1} \subset Q_0 \subset Q_1 \subset \dots \subset Q_{n-1} = \partial Q = |S|,$$

is called the coskeleton filtration of $|S|$ (see [12]).

The maximal pseudo-cell F_\emptyset of $P(S)$ is $P(S) \cong \text{Cone } |S|$, and $\partial F_\emptyset = |S|$. Note that ∂F_I can be identified with the barycentric subdivision of $\text{lk}_S I$. Face F_I is the cone over ∂F_I .

If S is non-pure, this construction makes sense as well, but the dimension of F_I may not be equal to $n - \dim I - 1$. So $P(S)$ is not a simple pseudo-cell complex if S is not pure.

For a general pseudo-cell complex Q there is a skeleton filtration

$$(3.2) \quad Q_0 \subset Q_1 \subset \dots \subset Q_{n-1} = \partial Q \subset Q_n = Q$$

and the corresponding spectral sequences in homology and cohomology are:

$$(3.3) \quad {}^Q E_{p,q}^1 = H_{p+q}(Q_p, Q_{p-1}) \Rightarrow H_{p+q}(Q), \quad d_Q^r: {}^Q E_{*,*}^r \rightarrow {}^Q E_{*-r, *+r-1}^r$$

$$(3.4) \quad {}^Q E_1^{p,q} = H^{p+q}(Q_p, Q_{p-1}) \Rightarrow H^{p+q}(Q) \quad (d_Q)_r: {}^Q E_r^{*,*} \rightarrow {}^Q E_r^{*+r, *-r+1}.$$

In the following only homological case is considered; the cohomological case being completely parallel.

Similar to ordinary cell complexes the first term of the spectral sequence is described as a sum:

$$H_{p+q}(Q_p, Q_{p-1}) \cong \bigoplus_{\dim F=p} H_{p+q}(F, \partial F).$$

The differential d_Q^1 is the sum over all pairs $I <_1 J \in S$ of the maps:

$$(3.5) \quad m_{I,J}^q: H_{q+\dim F_I}(F_I, \partial F_I) \rightarrow H_{q+\dim F_I-1}(\partial F_I) \rightarrow \\ \rightarrow H_{q+\dim F_I-1}(\partial F_I, \partial F_I \setminus F_J^\circ) \cong H_{q+\dim F_J}(F_J, \partial F_J),$$

where the last isomorphism is due to excision. Also consider the truncated spectral sequence

$$\partial^Q E_{p,q}^1 = H_{p+q}(Q_p, Q_{p-1}), p < n \Rightarrow H_{p+q}(\partial Q).$$

CONSTRUCTION 3.6. Given a sign convention on S_Q , for each q consider the sheaf \mathcal{H}_q on S_Q given by

$$\mathcal{H}_q(I) = H_{q+\dim F_I}(F_I, \partial F_I)$$

with restriction maps $\mathcal{H}_q(I <_1 J) = [J : I]m_{I,J}^q$. For general $I <_k J$ consider any saturated chain

$$(3.6) \quad I <_1 J_1 <_1 \dots <_1 J_{k-1} <_1 J$$

and set $\mathcal{H}_q(I <_k J)$ to be equal to the composition

$$\mathcal{H}_q(J_{k-1} <_1 J) \circ \dots \circ \mathcal{H}_q(I <_1 J_1).$$

LEMMA 3.7. *The map $\mathcal{H}_q(I <_k J)$ does not depend on a saturated chain (3.6).*

PROOF. The differential d_Q^1 satisfies $(d_Q^1)^2 = 0$, thus $m_{J',J}^q \circ m_{I,J'}^q + m_{J'',J}^q \circ m_{I,J''}^q = 0$. By combining this with (2.2) we prove that $\mathcal{H}_q(I <_2 J)$ is independent of a chain. In general, since $\{T \mid I \leq T \leq J\}$ is a boolean lattice, any two saturated chains between I and J are connected by a sequence of elementary flips $[J_k <_1 T_1 <_1 J_{k+2}] \rightsquigarrow [J_k <_1 T_2 <_1 J_{k+2}]$. \square

Thus the sheaves \mathcal{H}_q are well defined. These sheaves will be called the structure sheaves of Q . Consider also the truncated structure sheaves

$$\underline{\mathcal{H}}_q(I) = \begin{cases} \mathcal{H}_q(I) & \text{if } I \neq \emptyset, \\ 0, & \text{if } I = \emptyset. \end{cases}$$

COROLLARY 3.8. *The cochain complexes of structure sheaves coincide with ${}^Q E_{*,*}^1$ up to change of indices:*

$$({}^Q E_{*,q}^1, d_Q^1) \cong (C^{n-1-*}(\mathcal{H}_q), d), \quad (\partial^Q E_{*,q}^1, d_Q^1) \cong (C^{n-1-*}(\underline{\mathcal{H}}_q), d).$$

PROOF. Follows from the definition of the cochain complex of a sheaf. \square

REMARK 3.9. Let S be a pure simplicial poset of dimension $n - 1$ and $P(S)$ — its dual simple pseudo-cell complex. In this case there exists an isomorphism of sheaves

$$(3.7) \quad \underline{\mathcal{H}}_q \cong \mathcal{U}_{q+n-1},$$

where \mathcal{U}_* are the sheaves of local homology defined in example 2.10. Indeed, it can be shown that $H_i(S, S \setminus \text{lk}_S I) \cong H_{i-\dim I}(F_I, \partial F_I)$ and these isomorphisms can be chosen compatible with restriction maps. For simplicial complexes this fact is proved in [12, Sec.6.1]; the case of simplicial posets is rather similar. Note that \mathcal{H}_q depends on the sign convention while \mathcal{U} does not. There is a simple explanation: the isomorphism (3.7) itself depends on the orientations of simplices.

3.2. Buchsbaum pseudo-cell complexes.

DEFINITION 3.10. *A simple pseudo-cell complex Q of dimension n is called Buchsbaum if for any face $F_I \subset Q$, $I \neq \emptyset$ the following conditions hold:*

- (1) F_I is acyclic over \mathbb{Z} ;
- (2) $H_i(F_I, \partial F_I) = 0$ if $i \neq \dim F_I$.

Buchsbaum complex Q is called Cohen–Macaulay if these two conditions also hold for $I = \emptyset$.

The second condition in Buchsbaum case is equivalent to $\underline{\mathcal{H}}_q = 0$ for $q \neq 0$. Cohen–Macaulay case is equivalent to $\mathcal{H}_q = 0$ for $q \neq 0$. Obviously, Q is Buchsbaum if and only if all its proper faces are Cohen–Macaulay. Thus any face of dimension $p \geq 1$ has nonempty boundary of dimension $p - 1$. In particular, this implies S_Q is pure.

DEFINITION 3.11. *Buchsbaum pseudo-cell complex Q is called (\mathbb{k} -orientable) Buchsbaum manifold if $\underline{\mathcal{H}}_0$ is isomorphic to a constant sheaf \mathbb{k} .*

Note that this definition actually describes only the property of ∂Q not Q itself.

EXAMPLE 3.12. If Q is a nice compact manifold with corners in which every proper face is acyclic and orientable, then Q is a Buchsbaum pseudo-cell complex. Indeed, the second condition of 3.10 follows by Poincaré–Lefschetz duality. If, moreover, Q is orientable itself then Q is a Buchsbaum manifold (over all \mathbb{k}). Indeed, the restriction maps $\mathcal{H}_0(\emptyset \subset I)$ send the fundamental cycle $[Q] \in H_n(Q, \partial Q) \cong \mathbb{k}$ to fundamental cycles of proper faces, thus identifying $\underline{\mathcal{H}}_0$ with the constant sheaf \mathbb{k} . The choice of orientations establishing this identification is described in details in section 8.

EXAMPLE 3.13. Simplicial poset S is Buchsbaum (resp. Cohen–Macaulay) whenever $P(S)$ is a Buchsbaum (resp. Cohen–Macaulay) simple pseudo-cell complex. Indeed, any face of $P(S)$ is a cone, thus contractible. On the other hand, $H_i(F_I, \partial F_I) \cong H_i(\text{Cone} \mid \text{lk}_S I \mid, \mid \text{lk}_S I \mid) \cong \tilde{H}_{i-1}(\mid \text{lk}_S I \mid)$. Thus condition 2 in definition 3.10 is satisfied whenever $\tilde{H}_i(\text{lk}_S I) = 0$ for $i \neq n - 1 - |I|$. This is equivalent to Buchsbaumness (resp. Cohen–Macaulayness) of S by remark 2.3.

Poset S is a homology manifold if and only if $P(S)$ is a Buchsbaum manifold. This follows from remark 3.9. In particular, if $|S|$ is a closed orientable manifold then $P(S)$ is a Buchsbaum manifold.

In general, if Q is Buchsbaum, then its underlying poset S_Q is also Buchsbaum, see lemma 3.14 below.

In Buchsbaum (resp. Cohen–Macaulay) case the spectral sequence ${}^{\partial Q}E$ (resp. QE) collapses at the second page, thus

$$H^{n-1-p}(S_Q; \underline{\mathcal{H}}_0) \cong {}^{\partial Q}E_{p,0}^2 \xrightarrow{\cong} H_p(\partial Q), \quad \text{if } Q \text{ is Buchsbaum}$$

$$H^{n-1-p}(S_Q; \mathcal{H}_0) \cong {}^QE_{p,0}^2 \xrightarrow{\cong} H_p(Q), \quad \text{if } Q \text{ is Cohen–Macaulay}$$

In particular, if Q is a Buchsbaum manifold, then

$$(3.8) \quad H^{n-1-p}(S_Q) \cong H_p(\partial Q)$$

Let Q be a simple pseudo-cell complex, and $\emptyset \neq I \in S_Q$. The face F_I is a simple pseudo-cell complex itself, and $S_{F_I} = \text{lk}_S I$. The structure sheaves of F_I are the restrictions of \mathcal{H}_q to $\text{lk}_{S_Q} I \subset S_Q$. If Q is Buchsbaum, then F_I is Cohen–Macaulay, thus

$$(3.9) \quad H^k(\text{lk}_S I; \mathcal{H}_0) \xrightarrow{\cong} H_{\dim F_I - 1 - k}(F_I),$$

which is either \mathbb{k} (in case $k = \dim F_I - 1$) or 0 (otherwise), since F_I is acyclic.

3.3. Universality of posets. The aim of this subsection is to show that Buchsbaum pseudo-cell complex coincides up to homology with the underlying simplicial poset away from maximal cells. This was proved for nice manifolds with corners in [9] and essentially we follow the proof given there.

LEMMA 3.14.

(1)_n Let Q be Buchsbaum pseudo-cell complex of dimension n , S_Q — its underlying poset, and $P = P(S_Q)$ — simple pseudo-cell complex associated to S_Q (example 3.5), $\partial P = |S_Q|$. Then there exists a face-preserving map $\varphi: Q \rightarrow P$ which induces the identity isomorphism of posets and the isomorphism of the truncated spectral sequences $\varphi_*: {}^{\partial Q}E_{*,*}^r \xrightarrow{\cong} {}^{\partial P}E_{*,*}^r$ for $r \geq 1$.

(2)_n If Q is Cohen–Macaulay of dimension n , then φ induces the isomorphism of non-truncated spectral sequences $\varphi_*: {}^QE_{*,*}^r \xrightarrow{\cong} {}^PE_{*,*}^r$.

PROOF. The map φ is constructed inductively. 0-skeleta of Q and P are naturally identified. There always exists an extension of φ to higher-dimensional faces since all pseudo-cells of P are cones. The lemma is proved by the following scheme of induction: $(2)_{\leq n-1} \Rightarrow (1)_n \Rightarrow (2)_n$. The case $n = 0$ is clear. Let us prove $(1)_n \Rightarrow (2)_n$. The map φ induces the homomorphism of the long exact sequences:

$$\begin{array}{ccccccccc} \tilde{H}_*(\partial Q) & \longrightarrow & \tilde{H}_*(Q) & \longrightarrow & H_*(Q, \partial Q) & \longrightarrow & \tilde{H}_{*-1}(\partial Q) & \longrightarrow & \tilde{H}_{*-1}(Q) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \tilde{H}_*(\partial P) & \longrightarrow & \tilde{H}_*(P) & \longrightarrow & H_*(P, \partial P) & \longrightarrow & \tilde{H}_{*-1}(\partial P) & \longrightarrow & \tilde{H}_{*-1}(P) \end{array}$$

The maps $\tilde{H}_*(Q) \rightarrow \tilde{H}_*(P)$ are isomorphisms since both groups are trivial. The maps $\tilde{H}_*(\partial Q) \rightarrow \tilde{H}_*(\partial P)$ are isomorphisms by $(1)_n$, since ${}^{\partial Q}E \xrightarrow{\cong} H_*(\partial Q)$ and ${}^{\partial P}E \xrightarrow{\cong} H_*(\partial P)$. Five lemma shows that $\varphi_*: {}^Q E_{n,*}^1 \rightarrow {}^P E_{n,*}^1$ is an isomorphism as well. This imply $(2)_n$.

Now we prove $(2)_{\leq n-1} \Rightarrow (1)_n$. Let F_I be faces of Q and \tilde{F}_I — faces of P . All proper faces of Q are Cohen–Macaulay of dimension $\leq n-1$. Thus $(2)_{\leq n-1}$ implies isomorphisms $H_*(F_I, \partial F_I) \rightarrow H_*(\tilde{F}_I, \partial \tilde{F}_I)$ which sum together to the isomorphism $\varphi_*: {}^{\partial Q} E_{*,*}^1 \xrightarrow{\cong} {}^{\partial P} E_{*,*}^1$. \square

COROLLARY 3.15. *If Q is a Buchsbaum (resp. Cohen–Macaulay) pseudo-cell complex, then S_Q is a Buchsbaum (resp. Cohen–Macaulay) simplicial poset. If Q is a Buchsbaum manifold, then S_Q is a homology manifold.*

In particular, according to lemma 3.14, if Q is Buchsbaum, then ∂Q is homologous to $|S_Q| = \partial P(S_Q)$. So in the following we may not distinguish between their homology. If Q is Buchsbaum manifold, then (3.8) implies Poincare duality for ∂Q :

$$(3.10) \quad H^{n-1-p}(\partial Q) \cong H_p(\partial Q).$$

3.4. Structure of ${}^Q E_{*,*}^r$ in Buchsbaum case. Let $\delta_i: H_i(Q, \partial Q) \rightarrow H_{i-1}(\partial Q)$ be the connecting homomorphisms in the long exact sequence of the pair $(Q, \partial Q)$.

LEMMA 3.16. *The second term of ${}^Q E_{*,*}^*$ for Buchsbaum pseudo-cell complex Q is described as follows:*

$$(3.11) \quad {}^Q E_{p,q}^2 \cong \begin{cases} H_p(\partial Q), & \text{if } p \leq n-2, q=0, \\ \text{Coker } \delta_n, & \text{if } p=n-1, q=0, \\ \text{Ker } \delta_n, & \text{if } p=n, q=0, \\ H_{n+q}(Q, \partial Q), & \text{if } p=n, q<0, \\ 0, & \text{otherwise.} \end{cases}$$

PROOF. The first page of the non-truncated spectral sequence has the form

${}^Q E_{p,q}^1$		0	$n-2$	$n-1$	n		p
0	$C^{n-1}(S; \mathcal{H}_0)$	\dots	$C^1(S; \mathcal{H}_0)$	$C^0(S; \mathcal{H}_0)$	$C^{-1}(S; \mathcal{H}_0) = H_n(Q, \partial Q)$		
-1	0	\dots	0	0	$C^{-1}(S; \mathcal{H}_{-1}) = H_{n-1}(Q, \partial Q)$		
	\vdots		\vdots	\vdots	\vdots		
-n	0	\dots	0	0	$C^{-1}(S; \mathcal{H}_{-n}) = H_0(Q, \partial Q)$		
	q						

By the definition of Buchsbaum complex, ${}^Q E_{p,q}^2 = 0$ if $p < n$ and $q \neq 0$. Terms of the second page with $p \leq n - 2$ coincide with their non-truncated versions: ${}^Q E_{p,0}^2 = {}^{\partial Q} E_{p,0}^2 \cong H^{n-1-p}(S_Q; \underline{\mathcal{H}}_0) \cong H_p(\partial Q)$. For $p = n$, $q < 0$ the first differential vanishes, thus ${}^Q E_{n,q}^2 = {}^Q E_{n,q}^1 \cong H_{n+q}(Q, \partial Q)$. The only two cases that require further investigation are $(p, q) = (n - 1, 0)$ and $(n, 0)$. To describe these cases consider the short exact sequence of sheaves

$$(3.12) \quad 0 \rightarrow \underline{\mathcal{H}}_0 \rightarrow \mathcal{H}_0 \rightarrow \mathcal{H}_0/\underline{\mathcal{H}}_0 \rightarrow 0.$$

The quotient sheaf $\mathcal{H}_0/\underline{\mathcal{H}}_0$ is concentrated in degree -1 and its value on \emptyset is $H_n(Q, \partial Q)$. Sequence (3.12) induces the long exact sequence in cohomology (middle row):

$$\begin{array}{ccccccc} & & H_n(Q, \partial Q) & \xrightarrow{\delta_n} & H_{n-1}(\partial Q) & & \\ & & \cong \uparrow & & \cong \uparrow & & \\ 0 \longrightarrow & H^{-1}(S_Q; \mathcal{H}_0) & \longrightarrow & H^{-1}(S_Q; \mathcal{H}_0/\underline{\mathcal{H}}_0) & \longrightarrow & H^0(S_Q; \underline{\mathcal{H}}_0) & \longrightarrow H^0(S_Q; \mathcal{H}_0) \longrightarrow 0 \\ & \downarrow \cong & & & & \downarrow \cong & \\ & {}^Q E_{n,0}^2 & & & & {}^{\partial Q} E_{n-1,0}^2 & \longrightarrow {}^Q E_{n-1,0}^2 \end{array}$$

Thus ${}^Q E_{n,0}^2 \cong \text{Ker } \delta_n$ and ${}^Q E_{n-1,0}^2 \cong \text{Coker } \delta_n$. \square

In the situations like this, we call a spectral sequence \lrcorner -shaped. The only non-vanishing differentials in ${}^Q E^r$ for $r \geq 2$ are $d_r: {}^Q E_{n,1-r}^r \rightarrow {}^Q E_{n-r,0}^r$. They have pairwise different domains and targets, thus ${}^Q E_{p,q}^r \Rightarrow H_{p+q}(Q)$ folds in a long exact sequence, which is isomorphic to the long exact sequence of the pair $(Q, \partial Q)$:

$$(3.13) \quad \begin{array}{ccccccc} \dots & \longrightarrow & H_i(Q) & \longrightarrow & {}^Q E_{n,i-n}^{n+1-i} & \xrightarrow{d_Q^{n+1-i}} & {}^Q E_{i-1,0}^{n+1-i} \longrightarrow H_{i-1}(Q) \longrightarrow \dots \\ & & \parallel & & \uparrow \cong & & \uparrow \cong \\ & & & & {}^Q E_{n,i-n}^1 & & {}^Q E_{i-1,0}^2 \\ & & & & \parallel & & \downarrow \cong \\ \dots & \longrightarrow & H_i(Q) & \longrightarrow & H_i(Q, \partial Q) & \xrightarrow{\delta_i} & H_{i-1}(\partial Q) \longrightarrow H_{i-1}(Q) \longrightarrow \dots \end{array}$$

This gives a complete characterization of ${}^Q E$ in terms of the homological long exact sequence of the pair $(Q, \partial Q)$.

3.5. Artificial page ${}^Q E_{*,*}^{1+}$. In this subsection we formally introduce an additional term in the spectral sequence to make description of ${}^Q E_{*,*}^*$ more convenient

and uniform. The goal is to carry away δ_n (which appears in (3.11)) from the description of the page and treat it as one of higher differentials.

Let ${}^X E_{*,*}^{1+}$ be the collection of \mathbb{k} -modules defined by

$${}^Q E_{p,q}^{1+} \stackrel{\text{def}}{=} \begin{cases} \partial {}^Q E_{p,q}^2, & \text{if } p \leq n-1, \\ {}^Q E_{p,q}^1, & \text{if } p = n, \\ 0, & \text{otherwise.} \end{cases}$$

Let d_Q^{1-} be the differential of degree $(-1, 0)$ operating on $\bigoplus {}^Q E_{p,q}^1$ by:

$$d_Q^{1-} = \begin{cases} d_Q^1: {}^Q E_{p,q}^1 \rightarrow {}^Q E_{p-1,q}^1, & \text{if } p \leq n-1, \\ 0, & \text{otherwise} \end{cases}$$

It is easily seen that $H({}^Q E_{*,*}^1; d_Q^{1-})$ is isomorphic to ${}^Q E_{*,*}^{1+}$. Now consider the differential d_Q^{1+} of degree $(-1, 0)$ operating on $\bigoplus {}^Q E_{p,q}^{1+}$:

$$d_Q^{1+} = \begin{cases} 0, & \text{if } p \leq n-1; \\ {}^Q E_{n,q}^1 \xrightarrow{d_Q^1} {}^Q E_{n-1,q}^1 \longrightarrow {}^Q E_{n-1,q}^2, & \text{if } p = n. \end{cases}$$

Then ${}^Q E^2 \cong H({}^Q E^{1+}, d_Q^{1+})$. These considerations are shown on the diagram:

$$\begin{array}{ccccccc} & & {}^Q E^{1+} & & & & \\ & \nearrow d_Q^{1-} & & \nwarrow d_Q^{1+} & & & \\ {}^Q E^1 & & & & {}^Q E^2 & \xrightarrow{d_Q^2} & {}^Q E^3 \xrightarrow{d_Q^3} \dots \end{array}$$

in which the dotted arrows represent passing to homology. To summarize:

CLAIM 3.17. *There is a spectral sequence whose first page is $({}^Q E^{1+}, d_Q^{1+})$ and subsequent terms coincide with ${}^Q E^r$ for $r \geq 2$. Its nontrivial differentials for $r \geq 1$ are the maps*

$$d_Q^r: {}^Q E_{n,1-r}^r \rightarrow {}^Q E_{n-r,0}^r$$

which coincide up to isomorphism with

$$\delta_{n+1-r}: H_{n+1-r}(Q, \partial Q) \rightarrow H_{n-r}(\partial Q).$$

Thus the spectral sequence ${}^Q E_{*,*}^r$ for $r \geq 1+$ up to isomorphism has the form

$$\begin{array}{ccccccc}
 {}^Q E_{p,q}^{1+} & & & & & & \\
 \hline
 & H_0(\partial Q) & \cdots & H_{n-2}(\partial Q) & H_{n-1}(\partial Q) & \xleftarrow{d_Q^{1+} = \delta_n} & H_n(Q, \partial Q) \\
 & & & & & \nwarrow & \\
 & & & & & d_Q^2 = \delta_{n-1} & H_{n-1}(Q, \partial Q) \\
 & & & & & \nwarrow & \\
 & & & & & d_Q^n = \delta_1 & \vdots \\
 & & & & & \nwarrow & \\
 & & & & & & H_1(Q, \partial Q)
 \end{array}$$

4. Torus spaces over Buchsbaum pseudo-cell complexes

4.1. Preliminaries on torus maps. Let N be a nonnegative integer. Consider a compact torus $T^N = (S^1)^N$. The homology algebra $H_*(T^N; \mathbb{k})$ is the exterior algebra $\Lambda = \Lambda_{\mathbb{k}}[H_1(T^N)]$. Let $\Lambda^{(q)}$ denote the graded component of Λ of degree q , $\Lambda = \bigoplus_{q=0}^N \Lambda^{(q)}$, $\Lambda^{(q)} \cong H_q(T^N)$, $\dim \Lambda^{(q)} = \binom{N}{q}$.

If T^N acts on a space Z , then $H_*(Z)$ obtains the structure of Λ -module (i.e. two-sided Λ -module with property $a \cdot x = (-1)^{\deg a \deg x} x \cdot a$ for $a \in \Lambda$, $x \in H_*(Z)$); T^N -equivariant maps $f: Z_1 \rightarrow Z_2$ induce module homomorphisms $f_*: H_*(Z_1) \rightarrow H_*(Z_2)$; and equivariant filtrations induce spectral sequences with Λ -module structures.

CONSTRUCTION 4.1. Let \mathcal{T}_N be the set of all 1-dimensional toric subgroups of T^N . Let M be a finite subset of \mathcal{T}_N , i.e. a collection of subgroups $M = \{T_s^1, i_s: T_s^1 \hookrightarrow T^N\}$. Consider the homomorphism

$$i_M: T^M \rightarrow T^N, \quad T^M \stackrel{\text{def}}{=} \prod_M T_s^1, \quad i_M \stackrel{\text{def}}{=} \prod_M i_s.$$

DEFINITION 4.2. We say that the collection M of 1-dimensional subgroups satisfies $(*_\mathbb{k})$ -condition if the map $(i_M)_*: H_1(T^M; \mathbb{k}) \rightarrow H_1(T^N; \mathbb{k})$ is injective and splits.

If i_M itself is injective, then M satisfies $(*_\mathbb{k})$ for $\mathbb{k} = \mathbb{Z}$ and all fields. Moreover, $(*_\mathbb{Z})$ is equivalent to injectivity of i_M . Generally, $(*_\mathbb{k})$ implies that $\Gamma = \ker i_M$ is a finite subgroup of T^M .

For a set M satisfying $(*_\mathbb{k})$ consider the exact sequence

$$0 \rightarrow \Gamma \rightarrow T^M \xrightarrow{i_M} T^N \xrightarrow{\rho} G \rightarrow 0,$$

where $G = T^N / i_M(T^M)$ is isomorphic to a torus $T^{N-|M|}$.

LEMMA 4.3. *Let \mathcal{I}_M be the ideal of Λ generated by $i_M(H_1(T^M))$. Then there exists a unique map β which encloses the diagram*

$$\begin{array}{ccc} H_*(T^M) & \xrightarrow{\rho_*} & H_*(G) \\ \downarrow \cong & & \downarrow \beta \\ \Lambda & \xrightarrow{q} & \Lambda/\mathcal{I}_M \end{array}$$

and β is an isomorphism.

PROOF. We have $\rho_*: H_*(T^M) \rightarrow H_*(G) \cong \Lambda^*[H_1(G)]$. Map ρ is T^M -equivariant, thus ρ_* is a map of Λ -modules. Since $\rho_*((i_M)_*H_1(T^M)) = 0$, we have $\rho_*(\mathcal{I}_M) = 0$, thus ρ_* factors through the quotient module, $\rho_* = \beta \circ q$. Since ρ_* is surjective so is β . By $(*_\mathbb{k})$ -condition we have a split exact sequence

$$0 \rightarrow H_1(T^M) \rightarrow H_1(T^N) \rightarrow H_1(G) \rightarrow 0,$$

So far there is a section $\alpha: H_1(G) \rightarrow H_1(T^N)$ of the map ρ_* in degree 1. This section extends to $\tilde{\alpha}: H_*(G) = \Lambda^*[H_1(G)] \rightarrow \Lambda$, which is a section of ρ_* . Thus β is injective. \square

4.2. Principal torus bundles. Let $\rho: Y \rightarrow Q$ be a principal T^N -bundle over a simple pseudo-cell complex Q .

LEMMA 4.4. *If Q is Cohen–Macaulay, then Y is trivial. More precisely, there exists an isomorphism ξ :*

$$\begin{array}{ccc} Y & \xrightarrow{\xi} & Q \times T^N \\ & \searrow \rho & \swarrow \\ & Q & \end{array}$$

The induced isomorphism ξ_* identifies $H_*(Y, \partial Y)$ with $H_*(Q, \partial Q) \otimes \Lambda$ and $H_*(Y)$ with Λ .

PROOF. Principal T^N -bundles are classified by their Euler classes, sitting in $H^2(Q; \mathbb{Z}^N) = 0$ (recall that Q is acyclic over \mathbb{Z}). The second statement follows from the Künneth isomorphism. \square

For a general principal T^N -bundle $\rho: Y \rightarrow Q$ consider the filtration

$$(4.1) \quad \emptyset = Y_{-1} \subset Y_0 \subset Y_1 \subset \dots \subset Y_{n-1} \subset Y_n = Y,$$

where $Y_i = \rho^{-1}(Q_i)$. For each $I \in S_Q$ consider the subsets $Y_I = \rho^{-1}(F_I)$ and $\partial Y_I = \rho^{-1}(\partial F_I)$. In particular, $Y_\emptyset = Y$, $\partial Y = Y_{n-1}$.

Let ${}^Y E_{*,*}^*$ be the spectral sequence associated with filtration (4.1), i.e.:

$${}^Y E_{p,q}^1 \cong H_{p+q}(Y_p, Y_{p-1}) \Rightarrow H_{p+q}(Y), \quad d_Y^r: {}^Y E_{*,*}^r \rightarrow {}^Y E_{*-r, *+r-1}^r$$

and $H_{p+q}(Y_p, Y_{p-1}) \cong \bigoplus_{|I|=n-p} H_{p+q}(Y_I, \partial Y_I)$.

Similar to construction 3.6 we define the sheaf \mathcal{H}_q^Y on S_Q by setting

$$(4.2) \quad \mathcal{H}_q^Y(I) = H_{q+n-|I|}(Y_I, \partial Y_I).$$

The restriction maps coincide with the differential d_Y^1 up to incidence signs. Note that $\mathcal{H}^Y \stackrel{\text{def}}{=} \bigoplus_q \mathcal{H}_q^Y$ has a natural Λ -module structure induced by the torus action. The cochain complex of \mathcal{H}^Y coincides with the first page of ${}^Y E_{*,*}^*$ up to change of indices. As before, consider also the truncated spectral sequence:

$$\partial^Y E_{p,q}^1 \cong H_{p+q}(Y_p, Y_{p-1}), p < n \Rightarrow H_{p+q}(\partial Y),$$

and the truncated sheaf: $\underline{\mathcal{H}}_q^Y(\emptyset) = 0$, $\underline{\mathcal{H}}_q^Y(I) = \underline{\mathcal{H}}^Y(I)$ for $I \neq \emptyset$.

LEMMA 4.5. *If Q is Buchsbaum, then $\underline{\mathcal{H}}_q^Y \cong \underline{\mathcal{H}}_0 \otimes \mathcal{L}^{(q)}$, where $\mathcal{L}^{(q)}$ is a locally constant sheaf on S_Q valued by $\Lambda^{(q)}$.*

PROOF. All proper faces of Q are Cohen–Macaulay, thus lemma 4.4 applies. We have $H_q(Y_I, \partial Y_I) \cong H_0(F_I, \partial F_I) \otimes \Lambda^{(q)}$. For any $I < J$ there are two trivializations of Y_J : the restriction of ξ_I , and ξ_J itself:

$$\begin{array}{ccccc} & & Y_J & \hookrightarrow & Y_I \\ & \swarrow \xi_J & \downarrow \xi_I|_{Y_J} & & \downarrow \xi_I \\ F_J \times T^{\mathbb{N}} & \dashrightarrow & F_J \times T^{\mathbb{N}} & \hookrightarrow & F_I \times T^{\mathbb{N}} \end{array}$$

Transition maps $\xi_I|_{Y_J} \circ (\xi_J)^{-1}$ induce the isomorphisms in homology $\Lambda^{(q)} = H_q(F_J \times T^{\mathbb{N}}) \rightarrow H_q(F_I \times T^{\mathbb{N}}) = \Lambda^{(q)}$ which determine the restriction maps $\mathcal{L}^{(q)}(I \subset J)$. The locally constant sheaf $\mathcal{L}^{(q)}$ is thus defined, and the statement follows. \square

Denote $\mathcal{L} = \bigoplus_q \mathcal{L}^{(q)}$ — the graded sheaf on S_Q valued by $\Lambda = \bigoplus_q \Lambda^{(q)}$

REMARK 4.6. Our main example is the trivial bundle: $Y = Q \times T^{\mathbb{N}}$. In this case the whole spectral sequence ${}^Y E_{*,*}^*$ is isomorphic to ${}^Q E_{*,*}^* \otimes \Lambda$. For the structure sheaves we also have $\mathcal{H}_*^Y = \mathcal{H}_* \otimes \Lambda^*$. In particular the sheaf \mathcal{L} constructed in lemma 4.5 is globally trivial. By results of subsection 3.4, all terms and differentials of ${}^Y E_{*,*}^*$ are described explicitly. Nevertheless, several results of this paper remain valid in a general setting, thus are stated in full generality where it is possible.

REMARK 4.7. This construction is very similar to the construction of the sheaf of local fibers which appears in the Leray–Serre spectral sequence. But contrary to this general situation, here we construct not just a sheaf in a common topological sense, but a cellular sheaf supported on the given simplicial poset S_Q . Thus we prefer to provide all the details, even if they seem obvious to the specialists.

4.3. Torus spaces over simple pseudo-cell complexes. Recall, that $\mathcal{T}_{\mathbb{N}}$ denotes the set of all 1-dimensional toric subgroups of $T^{\mathbb{N}}$. Let Q be a simple pseudo-cell complex of dimension n , S_Q — its underlying simplicial poset and $\rho: Y \rightarrow Q$ — a principal $T^{\mathbb{N}}$ -bundle. There exists a general definition of a characteristic pair in the case of manifolds with locally standard actions, see [17, Def.4.2]. We do not review this definition here due to its complexity, but prefer to work in Buchsbaum setting, in which case many things simplify. If Q is Buchsbaum, then its proper faces F_I are Cohen–Macaulay, and according to lemma 4.4, there exist trivializations ξ_I which identify orbits over $x \in F_I$ with $T^{\mathbb{N}}$. If x belongs to several faces, then different trivializations give rise to the transition homeomorphisms $\mathrm{tr}_{I < J}: T^{\mathbb{N}} \rightarrow T^{\mathbb{N}}$, and at the global level some nontrivial twisting may occur. To give the definition of characteristic map, we need to distinguish between these different trivializations. Denote by $T^{\mathbb{N}}(I)$ the torus sitting over the face F_I (via trivialization of lemma 4.4) and let $\mathcal{T}_{\mathbb{N}}(I)$ be the set of 1-dimensional subtori of $T^{\mathbb{N}}(I)$. The map $\mathrm{tr}_{I < J}$ sends elements of $\mathcal{T}_{\mathbb{N}}(I)$ to $\mathcal{T}_{\mathbb{N}}(J)$ in an obvious way. One can think of $\mathcal{T}_{\mathbb{N}}(-)$ as a locally constant sheaf of sets on $S_Q \setminus \{\emptyset\}$.

DEFINITION 4.8. *A characteristic map λ is a collection of elements $\lambda(i) \in \mathcal{T}_{\mathbb{N}}(i)$ defined for each vertex $i \in \mathrm{Vert}(S_Q)$. This collection should satisfy the following condition: for any simplex $I \in S_Q$, $I \neq \emptyset$ with vertices i_1, \dots, i_k the set*

$$(4.3) \quad \{\mathrm{tr}_{i_1 < I} \lambda(i_1), \dots, \mathrm{tr}_{i_k < I} \lambda(i_k)\}$$

*satisfies $(*_k)$ condition in $T^{\mathbb{N}}(I)$.*

Clearly, a characteristic map exists only if $\mathbb{N} \geq n$. Let $T^{\lambda(I)}$ denote the subtorus of $T^{\mathbb{N}}(I)$ generated by 1-dimensional subgroups (4.3).

CONSTRUCTION 4.9 (Quotient construction). Consider the identification space:

$$(4.4) \quad X = Y / \sim,$$

where $y_1 \sim y_2$ if $\rho(y_1) = \rho(y_2) \in F_I^\circ$ for some $\emptyset \neq I \in S_Q$, and y_1, y_2 lie in the same $T^{\lambda(I)}$ -orbit.

There is a natural action of $T^{\mathbb{N}}$ on X coming from Y . The map $\mu: X \rightarrow Q$ is a projection to the orbit space $X/T^{\mathbb{N}} \cong Q$. The orbit $\mu^{-1}(b)$ over the point $b \in F_I^\circ \subset \partial Q$ is identified (via the trivializing homeomorphism) with $T^{\mathbb{N}}(I)/T^{\lambda(I)}$. This orbit has dimension $\mathbb{N} - \dim T^{\lambda(I)} = \mathbb{N} - |I| = \dim F_I + (\mathbb{N} - n)$. The preimages of points $b \in Q \setminus \partial Q$ are the full-dimensional orbits.

Filtration (4.1) descends to the filtration on X :

$$(4.5) \quad \emptyset = X_{-1} \subset X_0 \subset X_1 \subset \dots \subset X_{n-1} \subset X_n = X,$$

where $X_i = Y_i / \sim$ for $i \leq n$. In other words, X_i is the union of $(\leq i + \mathbb{N} - n)$ -dimensional orbits of the $T^{\mathbb{N}}$ -action. Thus $\dim X_i = 2i + \mathbb{N} - n$ for $i \leq n$.

Let ${}^X E_{*,*}^*$ be the spectral sequence associated with filtration (4.5):

$${}^X E_{p,q}^1 = H_{p+q}(X_p, X_{p-1}) \Rightarrow H_{p+q}(X), \quad d_X: {}^X E_{*,*}^r \rightarrow {}^X E_{*-r, *+r-1}^r.$$

The quotient map $f: Y \rightarrow X$ induces a morphism of spectral sequences $f_*^r: {}^Y E_{*,*}^r \rightarrow {}^X E_{*,*}^r$, which is a Λ -module homomorphism for each $r \geq 1$.

4.4. Structure of ${}^X E_{*,*}^1$. For each $I \in S_Q$ consider the subsets $X_I = Y_I/\sim$ and $\partial X_I = \partial Y_I/\sim$. As before, define the family of sheaves associated with filtration 4.5:

$$\mathcal{H}_q^X(I) = H_{q+n-|I|}(X_I, \partial X_I),$$

with the restriction maps equal to d_Y^1 up to incidence signs. These sheaves can be considered as a single sheaf \mathcal{H}^X graded by q . We have $({}^X E_{*,*}^1, d_X) \cong (C^{n-1-*}(S_Q, \mathcal{H}_q^X), d)$. There are natural morphisms of sheaves $f_*: \mathcal{H}_q^Y \rightarrow \mathcal{H}_q^X$ induced by the quotient map $f: Y \rightarrow X$, and the corresponding map of cochain complexes coincides with $f_*^1: {}^Y E_{*,*}^1 \rightarrow {}^X E_{*,*}^1$. Also consider the truncated versions: $\underline{\mathcal{H}}^X = \bigoplus_q \underline{\mathcal{H}}_q^X$ for which $\underline{\mathcal{H}}^X(\emptyset) = 0$.

REMARK 4.10. The map $f_*^1: H_*(Y, \partial Y) \rightarrow H_*(X, \partial X)$ is an isomorphism by excision since $X/\partial X \cong Y/\partial Y$.

Now we describe the truncated part of the sheaf \mathcal{H}^Y in algebraical terms. Let $I \in S_Q$ be a simplex and $i \leq I$ its vertex. Consider the element of exterior algebra $\omega_i \in \mathcal{L}(I)^{(1)} \cong \Lambda^{(1)}$ which is the image of the fundamental cycle of $\lambda(i) \cong T^1$ under the transition map $\text{tr}_{i \leq I}$:

$$(4.6) \quad \omega_i = (\text{tr}_{i \leq I})_* [\lambda(i)] \in \mathcal{L}(I)^{(1)}$$

Consider the subsheaf \mathcal{I} of \mathcal{L} whose value on a simplex I with vertices $\{i_1, \dots, i_k\} \neq \emptyset$ is:

$$(4.7) \quad \mathcal{I}(I) = (\omega_{i_1}, \dots, \omega_{i_k}) \subset \mathcal{L}(I),$$

— the ideal of the exterior algebra $\mathcal{L}(I) \cong \Lambda$ generated by linear forms. Also set $\mathcal{I}(\emptyset) = 0$. It is easily checked that $\mathcal{L}(I < J)\mathcal{I}(I) \subset \mathcal{I}(J)$, so \mathcal{I} is a well-defined subsheaf of \mathcal{L} .

LEMMA 4.11. *The map of sheaves $f_*: \underline{\mathcal{H}}_q^Y \rightarrow \underline{\mathcal{H}}_q^X$ is isomorphic to the quotient map of sheaves $\mathcal{H}_0 \otimes \mathcal{L}^{(q)} \rightarrow \mathcal{H}_0 \otimes (\mathcal{L}/\mathcal{I})^{(q)}$.*

PROOF. By lemma 4.4, $(Y_I, \partial Y_I) \rightarrow (F_I, \partial F_I)$ is equivalent to the trivial $T^{\mathbb{N}}$ -bundle $\xi_I: (Y_I, \partial Y_I) \cong (F_I, \partial F_I) \times T^{\mathbb{N}}(I)$. By construction of X , we have identifications

$$\xi'_I: (X_I, \partial X_I) \cong [(F_I, \partial F_I) \times T^{\mathbb{N}}(I)]/\sim.$$

By excision, the group $H_*([F_I \times T^{\mathbb{N}}(I)]/\sim, [\partial F_I \times T^{\mathbb{N}}(I)]/\sim)$ coincides with

$$H_*(F_I \times T^{\mathbb{N}}(I)/T^{\lambda(I)}, \partial F_I \times T^{\mathbb{N}}(I)/T^{\lambda(I)}) = H_*(F_I, \partial F_I) \otimes H_*(T^{\mathbb{N}}(I)/T^{\lambda(I)}).$$

The rest follows from lemma 4.3. \square

There is a short exact sequence of graded sheaves

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}/\mathcal{I} \longrightarrow 0$$

Tensoring it with \mathcal{H}_0 produces the short exact sequence

$$0 \longrightarrow \mathcal{H}_0 \otimes \mathcal{I}^{(q)} \longrightarrow \underline{\mathcal{H}}_q^Y \longrightarrow \underline{\mathcal{H}}_q^X \longrightarrow 0$$

according to lemma 4.11. The sheaf $\mathcal{H}_0 \otimes \mathcal{I}$ can also be considered as a subsheaf of non-truncated sheaf \mathcal{H}^Y .

LEMMA 4.12. *There is a short exact sequence of graded sheaves*

$$0 \rightarrow \mathcal{H}_0 \otimes \mathcal{I} \rightarrow \mathcal{H}^Y \rightarrow \mathcal{H}^X \rightarrow 0.$$

PROOF. Follows from the diagram

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & \mathcal{H}_0 \otimes \mathcal{I} & \hookrightarrow & \underline{\mathcal{H}}^Y & \longrightarrow & \underline{\mathcal{H}}^X \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathcal{H}_0 \otimes \mathcal{I} & \hookrightarrow & \mathcal{H}^Y & \longrightarrow & \mathcal{H}^X \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & \mathcal{H}^Y / \underline{\mathcal{H}}^Y & \xrightarrow{\cong} & \mathcal{H}^X / \underline{\mathcal{H}}^X \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

The lower sheaves are concentrated in $\emptyset \in S_Q$ and the graded isomorphism between them is due to remark 4.10. \square

4.5. Extra pages of ${}^Y E$ and ${}^X E$. To simplify further discussion we briefly sketch the formalism of additional pages of spectral sequences ${}^Y E$ and ${}^X E$, which extends considerations of subsection 3.5. Consider the following bigraded module:

$${}^Y E_{p,q}^{1+} = \begin{cases} \partial^Y E_{p,q}^2, & \text{if } p < n; \\ {}^Y E_{n,q}^1, & \text{if } p = n. \end{cases}$$

and define the differentials d_Y^{1-} on ${}^Y E^1$ and d_Y^{1+} on ${}^Y E^{1+}$ by

$$d_Y^{1-} = \begin{cases} d_Y^1, & \text{if } p < n; \\ 0, & \text{if } p = n. \end{cases} \quad d_Y^{1+} = \begin{cases} 0, & \text{if } p < n; \\ {}^Y E_{n,q}^1 \xrightarrow{d_Y^1} {}^Y E_{n-1,q}^1 \longrightarrow {}^Y E_{n-1,q}^2, & \text{if } p = n. \end{cases}$$

It is easily checked that ${}^Y E^{1+} \cong H({}^Y E^1, d_Y^{1-})$ and ${}^Y E^2 \cong H({}^Y E^{1+}, d_Y^{1+})$. The page ${}^X E^{1+}$ and the differentials d_X^{1-}, d_X^{1+} are defined similarly. The map $f_*^1: {}^Y E^1 \rightarrow {}^X E^1$ induces the map between the extra pages: $f_*^{1+}: {}^Y E^{1+} \rightarrow {}^X E^{1+}$.

5. Main results

5.1. Structure of ${}^X E_{*,*}^r$. The short exact sequence of lemma 4.12 generates the long exact sequence in sheaf cohomology:

$$(5.1) \quad \rightarrow H^{i-1}(S_Q; \mathcal{H}_0 \otimes \mathcal{I}^{(q)}) \rightarrow H^{i-1}(S_Q; \mathcal{H}_q^Y) \xrightarrow{f_*^2} H^{i-1}(S_Q; \mathcal{H}_q^X) \rightarrow H^i(S_Q; \mathcal{H}_0 \otimes \mathcal{I}^{(q)}) \rightarrow$$

The following lemma is the cornerstone of the whole work.

LEMMA 5.1 (Key Lemma). $H^i(S_Q; \mathcal{H}_0 \otimes \mathcal{I}^{(q)}) = 0$ if $i \leq n - 1 - q$.

The proof follows from a more general sheaf-theoretical fact and is postponed to section 6. In the following we simply write S instead of S_Q . By construction, ${}^Y E_{p,q}^2 \cong H^{n-1-p}(S; \mathcal{H}_q^Y)$ and ${}^X E_{p,q}^2 \cong H^{n-1-p}(S; \mathcal{H}_q^X)$. The Key lemma 5.1 and exact sequence (5.1) imply

LEMMA 5.2.

$$\begin{aligned} f_*^2: {}^Y E_{p,q}^2 &\rightarrow {}^X E_{p,q}^2 \text{ is an isomorphism if } p > q, \\ f_*^2: {}^Y E_{p,q}^2 &\rightarrow {}^X E_{p,q}^2 \text{ is injective if } p = q. \end{aligned}$$

In case $N = n$ this observation immediately describes ${}^X E_{*,*}^2$ in terms of ${}^Y E_{*,*}^2$. Under the notation

$$(d_Y)_{p,q}^r: {}^Y E_{p,q}^r \rightarrow {}^Y E_{p-r,q+r-1}^r, \quad (d_X)_{p,q}^r: {}^X E_{p,q}^r \rightarrow {}^X E_{p-r,q+r-1}^r$$

there holds

THEOREM 1. *Let Q be Buchsbaum pseudo-cell complex of dimension n , Y be a principal T^n -bundle over Q , $f: Y \rightarrow X = Y/\sim$ — the quotient construction, and $f_*^r: {}^Y E_{*,*}^r \rightarrow {}^X E_{*,*}^r$ — the induced map of homological spectral sequences associated with filtrations 4.1, 4.5. Then*

$$f_*^2: {}^Y E_{p,q}^2 \rightarrow {}^X E_{p,q}^2 \text{ is } \begin{cases} \text{an isomorphism if } q < p \text{ or } q = p = n, \\ \text{injective if } q = p < n, \end{cases}$$

and ${}^X E_{p,q}^2 = 0$ if $q > p$. Higher differentials of ${}^X E_{*,*}^*$ thus have the form

$$(d_X)_{p,q}^r = \begin{cases} f_*^r \circ (d_Y)_{p,q}^r \circ (f_*^r)^{-1}, & \text{if } p - r \geq q + r - 1, \\ 0 & \text{otherwise,} \end{cases}$$

for $r \geq 2$.

If Y is a trivial T^n -bundle, then the structure of ${}^Y E_{*,*}^* \cong {}^Q E_{*,*}^* \otimes \Lambda$ is described completely by subsection 3.4. In this case almost all the terms of ${}^X E_{*,*}^*$ are described explicitly.

THEOREM 2. *In the notation of Theorem 1 suppose $Y = Q \times T^n$. Let $\Lambda^{(q)} = H_q(T^n)$ and $\delta_i: H_i(Q, \partial Q) \rightarrow H_{i-1}(\partial Q)$ be the connecting homomorphisms. Then*

$$(5.2) \quad {}^X E_{p,q}^2 \cong \begin{cases} H_p(\partial Q) \otimes \Lambda^{(q)}, & \text{if } q < p \leq n-2; \\ \text{Coker } \delta_n \otimes \Lambda^{(q)}, & \text{if } q < p = n-1; \\ \text{Ker } \delta_n \otimes \Lambda^{(q)} \oplus \left(\bigoplus_{\substack{q_1+q_2=n+q \\ q_1 < n}} H_{q_1}(Q, \partial Q) \otimes \Lambda^{(q_2)} \right), & \text{if } q < p = n; \\ H_n(Q, \partial Q) \otimes \Lambda^{(n)}, & \text{if } q = p = n; \\ 0, & \text{if } q > p. \end{cases}$$

The maps $f_*^2: H_q(\partial Q) \otimes \Lambda^{(q)} \hookrightarrow {}^X E_{q,q}^2$ are injective for $q < n-1$. Higher differentials for $r \geq 2$ are the following:

$$d_X^r \cong \begin{cases} \begin{matrix} {}^X E_{n, q_1+q_2-n}^* & {}^X E_{q_1-1, q_2}^* \\ \delta_{q_1} \otimes \text{id}_\Lambda: H_{q_1}(Q, \partial Q) \otimes \Lambda^{(q_2)} \rightarrow H_{q_1-1}(\partial Q) \otimes \Lambda^{(q_2)}, \end{matrix} & \text{if } r = n - q_1 + 1, q_1 - 1 > q_2; \\ f_*^2 \circ (\delta_{q_1} \otimes \text{id}_\Lambda): H_{q_1}(Q, \partial Q) \otimes \Lambda^{(q_2)} \rightarrow H_{q_1-1}(\partial Q) \otimes \Lambda^{(q_2)} \hookrightarrow {}^X E_{q_1-1, q_1-1}^2, & \text{if } r = n - q_1 + 1, q_1 - 1 = q_2; \\ 0, & \text{otherwise.} \end{cases}$$

Using the formalism of extra pages introduced in subsection 4.5, Theorem 2 can be restated in a more convenient and concise form

STATEMENT 5.3. *There exists a spectral sequence whose first term is*

$$(5.3) \quad {}^X E_{p,q}^{1+} \cong \begin{cases} H_p(\partial Q) \otimes \Lambda^{(q)}, & \text{if } q < p < n; \\ \bigoplus_{q_1+q_2=q+n} H_{q_1}(Q, \partial Q) \otimes \Lambda^{(q_2)}, & \text{if } p = n; \\ 0, & \text{if } q > p; \end{cases}$$

and subsequent terms coincide with ${}^XE_{*,*}^r$ for $r \geq 2$. There exist injective maps $f_*^{1+} : H_q(\partial Q) \otimes \Lambda^{(q)} \hookrightarrow {}^XE_{q,q}^2$ for $q < n$. Differentials for $r \geq 1$ have the form

$$d_X^r \cong \begin{cases} {}^{XE_{n,q_1+q_2-n}^*} & {}^{XE_{q_1-1,q_2}^*} \\ \delta_{q_1} \otimes \text{id}_\Lambda : H_{q_1}(Q, \partial Q) \otimes \Lambda^{(q_2)} \rightarrow H_{q_1-1}(\partial Q) \otimes \Lambda^{(q_2)}, & \text{if } r = n - q_1 + 1, q_1 - 1 > q_2; \\ f_*^{1+} \circ (\delta_{q_1} \otimes \text{id}_\Lambda) : H_{q_1}(Q, \partial Q) \otimes \Lambda^{(q_2)} \rightarrow H_{q_1-1}(\partial Q) \otimes \Lambda^{(q_2)} \hookrightarrow {}^{XE_{q_1-1,q_1-1}^2}, & \text{if } r = n - q_1 + 1, q_1 - 1 = q_2; \\ 0, & \text{otherwise.} \end{cases}$$

Note that the terms ${}^XE_{q,q}^*$ for $q < n$ are not mentioned in the lists (5.2), (5.3). Let us call $\bigoplus_{q < n} {}^XE_{q,q}^{1+}$ the *border* of ${}^XE_{*,*}^*$. This name is due to the fact that all entries above the border vanish: ${}^XE_{p,q}^* = 0$ for $q > p$.

Denote $\dim_{\mathbb{k}} \tilde{H}_p(S) = \dim_{\mathbb{k}} \tilde{H}_p(\partial Q)$ by $\tilde{b}_p(S)$ for $p < n$. The ranks of the border components are described as follows:

THEOREM 3. *In the notation of Theorem 2 and statement 5.3*

$$\dim {}^XE_{q,q}^{1+} = h_q(S) + \binom{n}{q} \sum_{p=0}^q (-1)^{p+q} \tilde{b}_p(S)$$

for $q \leq n-1$, where $h_q(S)$ are the h -numbers of the simplicial poset S .

THEOREM 4.

(1) *Let Q be a Buchsbaum manifold over \mathbb{k} . Then $\dim {}^XE_{q,q}^{1+} = h'_{n-q}(S)$ for $q \leq n-2$ and ${}^XE_{n-1,n-1}^{1+} = h'_1(S) + n$.*

(2) *Let Q be Buchsbaum manifold such that $H_n(Q, \partial Q) \cong \mathbb{k}$ and $\delta_n : H_n(Q, \partial Q) \rightarrow H_{n-1}(\partial Q)$ is injective. Then $\dim {}^XE_{q,q}^2 = h'_{n-q}(S)$ for $0 \leq q \leq n$.*

The definitions of h -, h' - and h'' -vectors and the proof of Theorems 3, 4 and 5 are gathered in section 7. Note that it is sufficient to prove Theorem 3 and the first part of Theorem 4 in the case $Q = P(S)$. Indeed, by definition, ${}^XE_{q,q}^{1+} = {}^{\partial X}E_{q,q}^2$ for $q \leq n-1$, and there exists a map $(Q \times T^n)/\sim \rightarrow (P(S_Q) \times T^n)/\sim$ which covers the map φ of lemma 3.14 and induces the isomorphism of corresponding truncated spectral sequences.

In the cone case the border components can be described explicitly up to ∞ -term.

THEOREM 5. *Let S be a Buchsbaum poset, $Q = P(S)$, $Y = Q \times T^n$, $X = Y/\sim$, ${}^XE_{p,q}^r \Rightarrow H_{p+q}(X)$ — the homological spectral sequence associated with filtration (4.5). Then*

$$\dim {}^XE_{q,q}^\infty = h''_q(S)$$

for $0 \leq q \leq n$.

COROLLARY 5.4. *If S is Buchsbaum, then $h_i''(S) \geq 0$.*

PROOF. For any S there exists a characteristic map over \mathbb{Q} . Thus there exists a space $X = (P(S) \times T^n)/\sim$ and Theorem 5 applies. \square

5.2. Homology of X . Theorem 2 implies the additional grading on $H_*(S)$ — the one given by the degrees of exterior forms. It is convenient to work with this double grading.

CONSTRUCTION 5.5. Suppose $Y = Q \times T^n$. For $j \in [0, n]$ consider the \neg -shaped spectral sequence

$${}_j^Y E_{*,*}^r = {}^Q E_{*,*}^r \otimes \Lambda^{(j)}.$$

Clearly, ${}_j^Y E_{*,*}^r = \bigoplus_{j=0}^n {}_j^Y E_{*,*}^r$ and ${}_j^Y E_{p,q}^* \Rightarrow H_{p+q-j,j}(Y) \stackrel{\text{def}}{=} H_{p+q-j}(Q) \otimes \Lambda^{(j)}$. In particular,

$${}_j^Y E_{*,*}^{1+} = \bigoplus_{p < n} \left({}^Q E_{p,0}^{1+} \otimes \Lambda^{(j)} \right) \oplus \bigoplus_q \left({}^Q E_{n,q}^{1+} \otimes \Lambda^{(j)} \right)$$

Consider the corresponding \neg -shaped spectral subsequences in ${}^X E_{*,*}^*$. Start with the \mathbb{k} -modules:

$${}_j^X E_{*,*}^{1+} = \bigoplus_{p < n} {}^X E_{p,j}^{1+} \oplus \bigoplus_q f_*^{1+} \left({}^Q E_{n,q}^{1+} \otimes \Lambda^{(q)} \right).$$

By statement 5.3, all the differentials of ${}^X E_{*,*}^*$ preserve ${}_j^X E_{*,*}^*$, thus the spectral subsequences ${}_j^X E_{*,*}^r$ are well defined, and ${}_j^X E_{*,*}^r = \bigoplus_{j=0}^n {}_j^X E_{*,*}^r$. Let $H_{i,j}(X)$ be the family of subgroups of $H_*(X)$ such that ${}_j^X E_{p,q}^r \Rightarrow H_{p+q-j,j}(X)$. Then

$$H_k(X) = \bigoplus_{i+j=k} H_{i,j}(X)$$

and the map $f_*: H_*(Y) \rightarrow H_*(X)$ sends $H_{i,j}(Y) \cong H_i(Q) \otimes \Lambda^{(j)}$ to $H_{i,j}(X)$. The map $f_*^r: {}^Y E^r \rightarrow {}^X E^r$ sends ${}_j^Y E^r$ to ${}_j^X E^r$ for each $j \in \{0, \dots, n\}$ and we have commutative squares:

$$\begin{array}{ccc} {}_j^Y E_{p,q}^r & \Longrightarrow & H_{p+q-j,j}(Y) \\ \downarrow f_*^r & & \downarrow f_* \\ {}_j^X E_{p,q}^r & \Longrightarrow & H_{p+q-j,j}(X) \end{array}$$

PROPOSITION 5.6.

- (1) *If $i > j$, then $f_*: H_{i,j}(Y) \rightarrow H_{i,j}(X)$ is an isomorphism. In particular, $H_{i,j}(X) \cong H_i(Q) \otimes \Lambda^{(j)}$.*
- (2) *If $i < j$, then there exists an isomorphism $H_{i,j}(X) \cong H_i(Q, \partial Q) \otimes \Lambda^{(j)}$.*

(3) In case $i = j < n$, the module $H_{i,i}(X)$ fits in the exact sequence

$$0 \rightarrow {}^X E_{i,i}^\infty \rightarrow H_{i,i}(X) \rightarrow H_i(Q, \partial Q) \otimes \Lambda^{(i)} \rightarrow 0,$$

or, equivalently,

$$0 \rightarrow \text{Im } \delta_{i+1} \otimes \Lambda^{(i)} \rightarrow {}^X E_{i,i}^{1+} \rightarrow H_{i,i}(X) \rightarrow H_i(Q, \partial Q) \otimes \Lambda^{(i)} \rightarrow 0$$

(4) If $i = j = n$, then $H_{n,n}(X) = {}^X E_{n,n}^\infty = {}^X E_{n,n}^1$.

PROOF. According to statement 5.3

$$(5.4) \quad f_*^{1+}: {}^Y E_{i,q}^{1+} \rightarrow {}^X E_{i,q}^{1+} \text{ is } \begin{cases} \text{the isomorphism if } i > j \text{ or } i = j = n; \\ \text{injective if } i = j. \end{cases}$$

For each j both spectral sequences ${}^Y E$ and ${}^X E$ are \sqcap -shaped, thus fold in the long exact sequences:

$$(5.5) \quad \begin{array}{ccccccc} \dots & \longrightarrow & {}^Y E_{i,j}^{1+} & \longrightarrow & H_{i,j}(Y) & \longrightarrow & {}^Y E_{n,i-n+j}^{1+} \xrightarrow{d_Y^{n-i+1}} {}^Y E_{i-1,j}^{1+} \longrightarrow \dots \\ & & \downarrow f_* & & \downarrow f_* & & \cong \downarrow f_* \\ \dots & \longrightarrow & {}^X E_{i,j}^{1+} & \longrightarrow & H_{i,j}(X) & \longrightarrow & {}^X E_{n,i-n+j}^{1+} \xrightarrow{d_X^{n-i+1}} {}^X E_{i-1,j}^{1+} \longrightarrow \dots \end{array}$$

Applying five lemma in the case $i > j$ proves (1). For $i < j$, the groups ${}^X E_{i,j}^{1+}$, ${}^X E_{i-1,j}^{1+}$ vanish by dimensional reasons thus $H_{i,j}(X) \cong {}^X E_{n,i-n+j}^{1+} \cong {}^Y E_{n,i-n+j}^{1+} \cong H_i(Q, \partial Q) \otimes \Lambda^{(j)}$. Case $i = j$ also follows from (5.5) by a simple diagram chase. \square

In the manifold case proposition 5.6 reveals a bigraded duality. If Q is a nice manifold with corners, $Y = Q \times T^n$ and λ is a characteristic map over \mathbb{Z} , then X is a manifold with locally standard torus action. In this case Poincare duality respects the double grading.

PROPOSITION 5.7. *If $X = (Q \times T^n)/\sim$ is a manifold with locally standard torus action and \mathbb{k} is a field, then $H_{i,j}(X) \cong H_{n-i,n-j}(X)$.*

PROOF. If $i < j$, then $H_{i,j}(X) \cong H_i(Q, \partial Q) \otimes \Lambda^{(j)} \cong H_{n-i}(Q) \otimes \Lambda^{(n-j)} \cong H_{n-i,n-j}(X)$, since $H_i(Q, \partial Q) \cong H_{n-i}(Q)$ by the Poincare–Lefschetz duality and $H_j(T^n) \cong H_{n-j}(T^n)$ by the Poincare duality for the torus. The remaining isomorphism $H_{i,i}(X) \cong H_{n-i,n-i}(X)$ now follows from the ordinary Poincare duality for X . \square

REMARK 5.8. If the space $X = (Q \times T^n)/\sim$ is constructed from a manifold with corners $(Q, \partial Q)$ using characteristic map over \mathbb{Q} (i.e. X is a toric orbifold), then proposition 5.7 still holds over \mathbb{Q} .

6. Duality between certain cellular sheaves and cosheaves

6.1. Proof of the Key lemma. In this section we prove lemma 5.1. First recall the setting.

- Q : a Buchsbaum pseudo-cell complex with the underlying simplicial poset $S = S_Q$ (this poset is Buchsbaum itself by corollary 3.15).
- \mathcal{H}_0 : the structure sheaf on S ; $\mathcal{H}_0(J) = H_{\dim F_J}(F_J, \partial F_J)$.
- \mathcal{L} : a locally constant graded sheaf on S valued by exterior algebra $\Lambda = H_*(T^{\mathbb{N}})$. This sheaf is associated in a natural way to a principal $T^{\mathbb{N}}$ -bundle over Q , $\mathcal{L}(J) = H_*(T^{\mathbb{N}}(J))$ for $J \neq \emptyset$ and $\mathcal{L}(\emptyset) = 0$. By inverting all restriction maps we obtain the cosheaf $\hat{\mathcal{L}}$.
- λ : a characteristic map over S . It determines T^1 -subgroup $\text{tr}_{i \leq J}(\lambda(i)) \subset T^{\mathbb{N}}(J)$ for each simplex J with vertex i . The homology class of this subgroup is denoted $\omega_i \in \mathcal{L}(I)$ (see (4.6)). Note that the restriction isomorphism $\mathcal{L}(J_1 < J_2)$ sends $\omega_i \in \mathcal{L}(J_1)$ to $\omega_i \in \mathcal{L}(J_2)$. Thus we simply write ω_i for all such elements since the ambient exterior algebra will be clear from the context.
- \mathcal{I} : the sheaf of ideals, associated to λ . The value of \mathcal{I} on a simplex $J \neq \emptyset$ with vertices $\{i_1, \dots, i_k\}$ is the ideal $\mathcal{I}(J) = (\omega_{i_1}, \dots, \omega_{i_k}) \subset \mathcal{L}(J)$. Clearly, \mathcal{I} is a graded subsheaf of \mathcal{L} .

We now introduce another type of ideals.

CONSTRUCTION 6.1. Let $J = \{i_1, \dots, i_k\}$ be a nonempty subset of vertices of simplex $I \in S$. Consider the element $\pi_J \in \mathcal{L}(I)$, $\pi_J = \bigwedge_{i \in J} \omega_i$. By the definition of characteristic map, the elements ω_i are linearly independent, thus π_J is a non-zero $|J|$ -form. Let $\Pi_J \subset \mathcal{L}(I)$ be the principal ideal generated by π_J . The restriction maps $\mathcal{L}(I < I')$ identify $\Pi_J \subset \mathcal{L}(I)$ with $\Pi_J \subset \mathcal{L}(I')$.

In particular, when J is the whole set of vertices of a simplex $I \neq \emptyset$ we define $\hat{\Pi}(I) \stackrel{\text{def}}{=} \Pi_I \subset \hat{\mathcal{L}}(I)$. If $I' < I$, then the corestriction map $\hat{\mathcal{L}}(I > I') = \mathcal{L}(I' < I)^{-1}$ injects $\hat{\Pi}(I)$ into $\hat{\Pi}(I')$, since $\hat{\mathcal{L}}(I > I')\pi_I$ is divisible by $\pi_{I'}$. Thus $\hat{\Pi}$ is a well-defined graded subcosheaf of $\hat{\mathcal{L}}$. Formally set $\hat{\Pi}(\emptyset) = 0$.

THEOREM 6. *For Buchsbaum pseudo-cell complex Q and $S = S_Q$ there exists an isomorphism $H^k(S; \mathcal{H}_0 \otimes \mathcal{I}) \cong H_{n-1-k}(S; \hat{\Pi})$ which respects the gradings of \mathcal{I} and $\hat{\Pi}$.*

Before giving a proof let us deduce the Key lemma. We need to show that $H^i(S; \mathcal{H}_0 \otimes \mathcal{I}^{(q)}) = 0$ for $i \leq n - 1 - q$. According to Theorem 6 this is equivalent to

LEMMA 6.2. $H_i(S; \hat{\Pi}^{(q)}) = 0$ for $i \geq q$.

PROOF. The ideal $\hat{\Pi}(I) = \Pi_I$ is generated by the element π_I of degree $|I| = \dim I + 1$. Thus $\Pi_I^{(q)} = 0$ for $q \leq \dim I$. Hence the corresponding part of the chain complex vanishes. \square

PROOF OF THEOREM 6. The idea of proof is the following. First we construct a resolution of sheaf $\mathcal{H}_0 \otimes \mathcal{I}$ whose terms are “almost acyclic”. By passing to cochain complexes this resolution generates a bicomplex C^\bullet_\bullet . By considering two standard spectral sequences for this bicomplex we prove that both $H^k(S; \mathcal{H}_0 \otimes \mathcal{I})$ and $H_{n-1-k}(S; \hat{\Pi})$ are isomorphic to the cohomology of the totalization C^\bullet_{Tot} .

For each $\emptyset \neq I \in S$ consider the sheaf $\mathcal{R}_I = \mathcal{H}_0 \otimes \mathbb{I}^{\Pi_I}$ (see examples 2.8 and 2.10), i.e.:

$$\mathcal{R}_I(J) = \begin{cases} \mathcal{H}_0(J) \otimes \Pi_I, & \text{if } I \leq J; \\ 0 & \text{otherwise.} \end{cases}$$

The sheaf \mathcal{R}_I is graded by degrees of exterior forms: $\mathcal{R}_I = \bigoplus_q \mathcal{R}_I^{(q)}$. Since $I > I'$ implies $\Pi_I \subset \Pi_{I'}$, and $i \in \text{Vert}(S)$, $i \leq J$ implies $\Pi_i \subset \mathcal{I}(J)$, there exist natural injective maps of sheaves:

$$\theta_{I>I'}: \mathcal{R}_I \hookrightarrow \mathcal{R}_{I'},$$

and

$$\eta_i: \mathcal{R}_i \hookrightarrow \mathcal{H}_0 \otimes \mathcal{I}.$$

For each $k \geq 0$ consider the sheaf

$$\mathcal{R}_{-k} = \bigoplus_{\dim I=k} \mathcal{R}_I,$$

These sheaves can be arranged in the sequence

$$(6.1) \quad \mathcal{R}_\bullet: \quad \dots \longrightarrow \mathcal{R}_{-2} \xrightarrow{d_H} \mathcal{R}_{-1} \xrightarrow{d_H} \mathcal{R}_0 \xrightarrow{\eta} \mathcal{H}_0 \otimes \mathcal{I} \longrightarrow 0,$$

where $d_H = \bigoplus_{I>I'} [I : I'] \theta_{I>I'}$ and $\eta = \bigoplus_{i \in \text{Vert}(S)} \eta_i$. By the standard argument involving incidence numbers $[I : I']$ one shows that (6.1) is a differential complex of sheaves. Moreover,

LEMMA 6.3. *The sequence \mathcal{R}_\bullet is exact.*

PROOF. We should prove that the value of \mathcal{R}_\bullet at each $J \in S$ is exact. Since $\mathcal{R}_I(J) \neq 0$ only if $I \leq J$ the complex $\mathcal{R}_\bullet(J)$ has the form

$$(6.2) \quad \dots \longrightarrow \bigoplus_{I \leq J, |I|=2} \Pi_I \longrightarrow \bigoplus_{I \leq J, |I|=1} \Pi_I \longrightarrow \mathcal{I}(J) \longrightarrow 0,$$

tensored with $\mathcal{H}_0(J)$. Without loss of generality we forget about $\mathcal{H}_0(J)$. Maps in (6.2) are given by inclusions of sub-ideals (rectified by incidence signs). This looks very similar to the Taylor resolution of monomial ideal in commutative polynomial ring, but our situation is a bit different, since Π_I are not free modules over Λ . Anyway, the proof is similar to commutative case: exactness of (6.2) follows from inclusion-exclusion principle. To make things precise (and also to tackle the case $\mathbb{k} = \mathbb{Z}$) we proceed as follows.

By $(*_\mathbb{k})$ -condition, the subspace $\langle \omega_j \mid j \in J \rangle$ is a direct summand in $\mathcal{L}^{(1)}(J) \cong \mathbb{k}^n$. Let $\{\nu_1, \dots, \nu_n\}$ be such a basis of $\mathcal{L}^{(1)}(J)$, that its first $|J|$ vectors are identified

with ω_j , $j \in J$. We simply write J for $\{1, \dots, |J|\} \subseteq [n]$ by abuse of notation. The module Λ splits in the multidegree components: $\Lambda = \bigoplus_{A \subseteq [n]} \Lambda_A$, where Λ_A is a 1-dimensional \mathbb{k} -module generated by $\bigwedge_{i \in A} \nu_i$. All modules and maps in (6.2) respect this splitting. Thus (6.2) can be written as

$$\begin{aligned} \dots \longrightarrow \bigoplus_{I \subseteq J, |I|=2} \bigoplus_{A \supseteq I} \Lambda_A \longrightarrow \bigoplus_{I \subseteq J, |I|=1} \bigoplus_{A \supseteq I} \Lambda_A \longrightarrow \bigoplus_{A \cap J \neq \emptyset} \Lambda_A \longrightarrow 0, \\ \bigoplus_{A, A \cap J \neq \emptyset} \left(\dots \longrightarrow \bigoplus_{I \subseteq A \cap J, |I|=2} \Lambda_A \longrightarrow \bigoplus_{I \subseteq A \cap J, |I|=1} \Lambda_A \longrightarrow \Lambda_A \longrightarrow 0 \right) \end{aligned}$$

For each A the cohomology of the complex in brackets coincides with $\tilde{H}_*(\Delta_{A \cap J}; \Lambda_A) \cong \tilde{H}_*(\Delta_{A \cap J}; \mathbb{k})$, the reduced simplicial homology of a simplex on the set $A \cap J \neq \emptyset$. Thus homology vanishes. \square

By passing to cochains and forgetting the last term we get a complex of complexes (6.3)

$$C^\bullet_\bullet = C(S, \mathcal{R}_\bullet): \quad \dots \longrightarrow C^\bullet(S; \mathcal{R}_{-2}) \xrightarrow{d_H} C^\bullet(S; \mathcal{R}_{-1}) \xrightarrow{d_H} C^\bullet(S; \mathcal{R}_0) \longrightarrow 0,$$

whose horizontal cohomology vanishes except for the upmost right position. Let d_V be the “vertical” cohomology differential operating in each $C^\bullet(S; \mathcal{R}_k^{(q)})$. Then $d_H d_V = d_V d_H$. Thus C^\bullet_\bullet can be considered as a bicomplex $(C^\bullet_{\text{Tot}}, D)$:

$$C^\bullet_{\text{Tot}} = \bigoplus_i C^i_{\text{Tot}}, \quad C^i_{\text{Tot}} = \bigoplus_{k+l=i} C^l_k, \quad D = d_H + (-1)^k d_V.$$

There are two standard spectral sequences converging to $H^*(C^\bullet_{\text{Tot}}, D)$ [11, Ch.2.4]. The first one, horizontal:

$${}^H E_r^{*,*}, \quad d_r^H: {}^H E_r^{k,l} \rightarrow {}^H E_r^{k+1-r, l+r}$$

computes horizontal cohomology first, then vertical cohomology. The second, vertical,

$${}^V E_r^{*,*}, \quad d_r^V: {}^V E_r^{k,l} \rightarrow {}^V E_r^{k+r, l+1-r}$$

computes vertical cohomology first, then horizontal.

LEMMA 6.4. $H^k(C^\bullet_{\text{Tot}}, D) \cong H^k(S; \mathcal{H}_0 \otimes \mathcal{I})$.

PROOF. Consider the horizontal spectral sequence:

$$\begin{aligned} {}^H E_1^{k,l} = H^k(C^l(S, \mathcal{R}_k), d^H) &= \begin{cases} C^l(S; \mathcal{H}_0 \otimes \mathcal{I}), & k = 0; \\ 0, & \text{o.w.} \end{cases} \\ {}^H E_2^{k,l} &= \begin{cases} H^l(S; \mathcal{H}_0 \otimes \mathcal{I}), & \text{if } k = 0; \\ 0, & \text{o.w.} \end{cases} \end{aligned}$$

Spectral sequence ${}^H E_*^{*,*}$ collapses at the second term and the statement follows. \square

LEMMA 6.5. $H^k(C_{\text{Tot}}^\bullet, D) \cong H_{n-1-k}(S; \widehat{\Pi})$.

PROOF. Consider the vertical spectral sequence. It starts with

$${}^V E_1^{k,l} \cong H^l(S; \mathcal{R}_k) = \bigoplus_{\dim I = -k} H^l(S; \mathcal{R}_I).$$

Similar to example 2.9 we get

$$H^l(S; \mathcal{R}_I) = H^l(S; \mathcal{H}_0 \otimes {}^l I^{\Pi_I}) = H^{l-|I|}(\text{lk}_S I; \mathcal{H}_0|_{\text{lk}_I} \otimes \Pi_I)$$

The restriction of \mathcal{H}_0 to $\text{lk}_S I \subset S$ coincides with the structure sheaf of F_I and by (3.9) we have a collapsing

$$H^{l-|I|}(\text{lk}_S I; \mathcal{H}_0|_{\text{lk}_I} \otimes \Pi_I) \xrightarrow{\cong} H_{n-1-l}(F_I) \otimes \Pi_I$$

A proper face F_I is acyclic, thus

$$H^l(S; \mathcal{R}_I) \cong \begin{cases} \Pi_I, & \text{if } l = n-1, \\ 0, & \text{if } l \neq n-1. \end{cases}$$

The maps $\theta_{I>I'}$ induce the isomorphisms $H_*(F_I) \rightarrow H_*(F_{I'})$ which assemble in commutative squares

$$\begin{array}{ccc} H^{n-1}(S; \mathcal{R}_I) & \xrightarrow{\cong} & \Pi_I \\ \downarrow \theta_{I>I'}^* & & \downarrow \\ H^{n-1}(S; \mathcal{R}_{I'}) & \xrightarrow{\cong} & \Pi_{I'} \end{array}$$

Thus the first term of vertical spectral sequence is identified with the chain complex of cosheaf $\widehat{\Pi}$:

$${}^V E_1^{k,l} = \begin{cases} C_{-k}(S; \widehat{\Pi}), & \text{if } l = n-1; \\ 0, & \text{if } l \neq n-1. \end{cases} \quad {}^V E_2^{k,l} = \begin{cases} H_{-k}(S; \widehat{\Pi}), & \text{if } l = n-1; \\ 0, & \text{if } l \neq n-1. \end{cases}$$

The spectral sequence ${}^V E_*^{k,l} \Rightarrow H^{k+l}({}_q C_{\text{Tot}}^\bullet, D)$ collapses at the second page. Lemma proved. \square

Theorem 6 follows from lemmas 6.4 and 6.5. \square

6.2. Extending duality to exact sequences. Theorem 6 can be refined:

STATEMENT 6.6. *The short exact sequence of sheaves*

$$(6.4) \quad 0 \rightarrow \mathcal{H}_0 \otimes \mathcal{I} \rightarrow \mathcal{H}_0 \otimes \mathcal{L} \rightarrow \mathcal{H}_0 \otimes (\mathcal{L}/\mathcal{I}) \rightarrow 0$$

and the short exact sequence of cosheaves

$$0 \rightarrow \widehat{\Pi} \rightarrow \widehat{\mathcal{L}} \rightarrow \widehat{\mathcal{L}}/\widehat{\Pi} \rightarrow 0$$

induce isomorphic long exact sequences in (co)homology:

$$(6.5) \quad \begin{array}{ccccccc} H^i(S; \mathcal{H}_0 \otimes \mathcal{I}) & \longrightarrow & H^i(S; \mathcal{H}_0 \otimes \mathcal{L}) & \longrightarrow & H^i(S; \mathcal{H}_0 \otimes (\mathcal{L}/\mathcal{I})) & \longrightarrow & H^{i+1}(S; \mathcal{H}_0 \otimes \mathcal{I}) \\ \uparrow \cong & & \uparrow \cong & & \uparrow \cong & & \uparrow \cong \\ H_{n-1-i}(S; \hat{\Pi}) & \longrightarrow & H_{n-1-i}(S; \hat{\mathcal{L}}) & \longrightarrow & H_{n-1-i}(S; (\hat{\mathcal{L}}/\hat{\Pi})) & \longrightarrow & H^{n-2-i}(S; \hat{\Pi}) \end{array}$$

PROOF. The proof goes essentially the same as in Theorem 6. Denote sequence (6.4) by ${}_{seq}\mathcal{I}$. For each $\emptyset \neq I \in S$ consider the short exact sequence of sheaves:

$${}_{seq}R_I \quad : \quad 0 \rightarrow \mathcal{R}_I \rightarrow \mathcal{H}_0 \otimes \mathcal{L} \rightarrow \mathcal{H}_0 \otimes (\mathcal{L}/{}^{\mathbf{l}}I^{\Pi_I}) \rightarrow 0$$

and define

$${}_{seq}R_{-k} = \bigoplus_{\dim I=k} {}_{seq}R_I$$

One can view ${}_{seq}\mathcal{I}$, ${}_{seq}R_I$ and ${}_{seq}R_{-k}$ as the objects in a category of complexes. As before, we can form the sequence

$$(6.6) \quad {}_{seq}R_{\bullet} : \quad \dots \longrightarrow {}_{seq}R_{-2} \xrightarrow{d_H} {}_{seq}R_{-1} \xrightarrow{d_H} {}_{seq}R_0 \xrightarrow{\eta} {}_{seq}\mathcal{I} \longrightarrow 0,$$

which happens to be exact in all positions. This long exact sequence (after forgetting the last term) generates the bicomplex of short exact sequences (or the short exact sequence of bicomplexes) ${}_{seq}\mathcal{C}_{\bullet}^{\bullet}$. By taking totalization and considering standard spectral sequences we check that both rows in (6.5) are isomorphic to the long exact sequence of cohomology associated to $({}_{seq}\mathcal{C}_{\text{Tot}}^{\bullet}, D)$. \square

6.3. Remark on duality. In the manifold case (i.e. sheaf \mathcal{H}_0 is isomorphic to \mathbb{k}), the proof of Theorem 6 can be restated in more conceptual terms. In this case the cellular version of Verdier duality for manifolds [6, Th.12.3] asserts:

$$H^i(S; \mathcal{I}) = H_{n-1-i}(S; \mathcal{I}),$$

where the homology groups of a cellular sheaf are defined as homology of global sections of projective sheaf resolution [6, Def.11.29]. The sheaf $\mathcal{R}_I = \mathcal{H}_0 \otimes {}^{\mathbf{l}}I^{\Pi_I} \cong {}^{\mathbf{l}}I^{\Pi_I}$ is projective ([6, Sec.11.1.1]), thus (6.1) is actually a projective resolution. Due to the specific structure of this resolution, we have $H_*(S; \mathcal{I}) \cong H_*(S; \hat{\Pi})$.

7. Face vectors and ranks of border components

In this section we prove Theorems 3, 4 and 5.

7.1. Preliminaries on face vectors. First recall several standard definitions from combinatorial theory of simplicial complexes and posets.

CONSTRUCTION 7.1. Let S be a pure simplicial poset, $\dim S = n - 1$. Let $f_i(S) = \{I \in S \mid \dim I = i\}$, $f_{-1}(S) = 1$. The array $(f_{-1}, f_0, \dots, f_{n-1})$ is called the f -vector of S . We write f_i instead of $f_i(S)$ since S is clear from the context. Let $f_S(t)$ be the generating polynomial: $f_S(t) = \sum_{i \geq 0} f_{i-1} t^i$.

Define the h -numbers by the relation:

$$(7.1) \quad \sum_{i=0}^n h_i(S) t^i = \sum_{i=0}^n f_{i-1} t^i (1-t)^{n-i} = (1-t)^n f_S\left(\frac{t}{1-t}\right).$$

Let $b_i(S) = \dim H_i(S)$, $\tilde{b}_i(S) = \dim \tilde{H}_i(S)$, $\chi(S) = \sum_{i=0}^{n-1} (-1)^i b_i(S) = \sum_{i=0}^{n-1} (-1)^i f_i(S)$ and $\tilde{\chi}(S) = \sum_{i=0}^{n-1} \tilde{b}_i(S) = \chi(S) - 1$. Thus $f_S(-1) = 1 - \chi(S)$. Also note that $h_n(S) = (-1)^{n-1} \tilde{\chi}(S)$.

Define h' - and h'' -vectors by

$$(7.2) \quad h'_i = h_i + \binom{n}{i} \left(\sum_{j=1}^{i-1} (-1)^{i-j-1} \tilde{b}_{j-1}(S) \right) \text{ for } 0 \leq i \leq n;$$

$$(7.3) \quad h''_i = h'_i - \binom{n}{i} \tilde{b}_{i-1}(S) = h_i + \binom{n}{i} \left(\sum_{j=1}^i (-1)^{i-j-1} \tilde{b}_{j-1}(S) \right) \text{ for } 0 \leq i \leq n-1,$$

and $h''_n = h'_n$. Note that

$$(7.4) \quad h'_n = h_n + \sum_{j=0}^{n-1} (-1)^{n-j-1} \tilde{b}_{j-1}(S) = \tilde{b}_{n-1}(S).$$

STATEMENT 7.2 (Dehn–Sommerville relations). *If S is Buchsbaum and $\dim \mathcal{H}_0(I) = 1$ for each $I \neq \emptyset$, then*

$$(7.5) \quad h_i = h_{n-i} + (-1)^i \binom{n}{i} (1 - (-1)^n - \chi(S)),$$

or, equivalently:

$$(7.6) \quad h_i = h_{n-i} + (-1)^i \binom{n}{i} (1 + (-1)^n \tilde{\chi}(S)),$$

If, moreover, S is a homology manifold, then $h''_i = h''_{n-i}$.

PROOF. The first statement can be found e.g. in [16] or [4, Thm.3.8.2]. Also see remark 7.5 below. The second then follows from the definition of h'' -vector and Poincare duality (3.8) $b_i(S) = b_{n-1-i}(S)$. \square

DEFINITION 7.3. *Let S be Buchsbaum. For $i \geq 0$ consider*

$$\hat{f}_i(S) = \sum_{I \in S, \dim I = i} \dim \tilde{H}_{n-1-|I|}(\mathrm{lk}_S I) = \sum_{I \in S, \dim I = i} \dim \mathcal{H}_0(I).$$

If S is a homology manifold, then $\hat{f}_i(S) = f_i(S)$. For general Buchsbaum complexes there is another formula connecting these quantities.

PROPOSITION 7.4. $f_S(t) = (1 - \chi(S)) + (-1)^n \sum_{k \geq 0} \hat{f}_k(S) \cdot (-t - 1)^{k+1}$.

PROOF. This follows from the general statement [8, Th.9.1], [4, Th.3.8.1], but we provide an independent proof for completeness. As stated in [1, Lm.3.7,3.8] for simplicial complexes (and also not difficult to prove for posets) $\frac{d}{dt} f_S(t) = \sum_{v \in \mathrm{Vert}(S)} f_{\mathrm{lk}_v}(t)$, and, more generally,

$$\left(\frac{d}{dt}\right)^k f_S(t) = k! \sum_{I \in S, |I|=k} f_{\mathrm{lk}_I}(t).$$

Thus for $k \geq 1$:

$$\begin{aligned} f_S^{(k)}(-1) &= k! \sum_{I \in S, |I|=k} (1 - \chi(\mathrm{lk}_S I)) = \\ &= k! \sum_{I \in S, |I|=k} (-1)^{n-|I|} \dim \tilde{H}_{n-|I|-1}(\mathrm{lk}_S I) = (-1)^{n-k} k! \hat{f}_{k-1}(S). \end{aligned}$$

Considering the Taylor expansion of $f_S(t)$ at -1 :

$$f_S(t) = f_S(-1) + \sum_{k \geq 1} \frac{1}{k!} f_S^{(k)}(-1) (t+1)^k = (1 - \chi(S)) + \sum_{k \geq 0} (-1)^{n-k-1} \hat{f}_k(S) \cdot (t+1)^{k+1},$$

finishes the proof. \square

REMARK 7.5. If S is a manifold, then proposition 7.4 implies

$$f_S(t) = (1 - (-1)^n - \chi(S)) + (-1)^n f_S(-t - 1),$$

which is an equivalent form of Dehn–Sommerville relations (7.5).

LEMMA 7.6. *For Buchsbaum poset S there holds*

$$\sum_{i=0}^n h_i t^i = (1 - t)^n (1 - \chi(S)) + \sum_{k \geq 0} \hat{f}_k(S) \cdot (t - 1)^{n-k-1}.$$

PROOF. Substitute $t/(1 - t)$ in proposition 7.4 and use (7.1). \square

The coefficients of t^i in lemma 7.6 give the relations

$$(7.7) \quad h_i(S) = (1 - \chi(S))(-1)^i \binom{n}{i} + \sum_{k \geq 0} (-1)^{n-k-i-1} \binom{n-k-1}{i} \hat{f}_k(S).$$

7.2. Ranks of ${}^XE_{*,*}^1$. Our goal is to compute the ranks of border groups $\dim {}^XE_{q,q}^{1+}$. The idea is very straightforward: statement 5.3 describes the ranks of all groups ${}^XE_{p,q}^{1+}$ except for $p = q$; and the terms ${}^XE_{p,q}^1$ are known as well; thus $\dim {}^XE_{q,q}^{1+}$ can be found by comparing Euler characteristics. Note that the terms with $p = n$ do not change when passing from ${}^XE^1$ to ${}^XE^{1+}$. Thus it is sufficient to perform calculations with the truncated sequence ${}^{\partial X}E$. By construction, ${}^XE_{p,q}^1 \cong {}^{\partial X}E_{p,q}^1 \cong C^{n-p-1}(S; \mathcal{H}_q^X)$ for $p < n$. Thus lemma 4.11 implies for $p < n$:

$$\dim {}^XE_{p,q}^1 = \dim {}^{\partial X}E_{p,q}^1 = \sum_{|I|=n-p} \dim \mathcal{H}_0(I) \cdot \dim(\Lambda/\mathcal{I}(I))^{(q)} = \binom{p}{q} \cdot \hat{f}_{n-p-1}(S).$$

Let χ_q^1 be the Euler characteristic of q -th row of ${}^{\partial X}E_{*,*}^1$:

$$(7.8) \quad \chi_q^1 = \sum_{p \leq n-1} (-1)^p \dim {}^{\partial X}E_{p,q}^1 = \sum_{p \leq n-1} (-1)^p \binom{p}{q} \hat{f}_{n-p-1}$$

LEMMA 7.7. *For $q \leq n-1$ there holds $\chi_q^1 = (\chi(S) - 1) \binom{n}{q} + (-1)^q h_q(S)$.*

PROOF. Substitute $i = q$ and $k = n - p - 1$ in (7.7) and combine with (7.8). \square

7.3. Ranks of ${}^XE_{*,*}^{1+}$. By construction of the extra page, ${}^XE_{p,q}^{1+} \cong {}^{\partial X}E_{p,q}^2$ for $p < n$. Let χ_q^2 be the Euler characteristic of q -th row of ${}^{\partial X}E_{*,*}^2$:

$$(7.9) \quad \chi_q^2 = \sum_p (-1)^p \dim {}^{\partial X}E_{p,q}^2.$$

Euler characteristics of first and second terms coincide: $\chi_q^2 = \chi_q^1$. By statement 5.3, $\dim {}^XE_{p,q}^{1+} = \binom{n}{q} b_p(S)$ for $q < p < n$. Lemma 7.7 yields

$$(-1)^q \dim {}^XE_{q,q}^{1+} + \sum_{p=q+1}^{n-1} (-1)^p \binom{n}{q} b_p(S) = (\chi(S) - 1) \binom{n}{q} + (-1)^q h_q.$$

By taking into account obvious relations between reduced and non-reduced Betti numbers and equality $\chi(S) = \sum_{p=0}^{n-1} b_p(S)$, this proves Theorem 3.

7.4. Manifold case. If X is a homology manifold, then Poincare duality $b_i(S) = b_{n-i}(S)$ and Dehn–Sommerville relations (7.6) imply

$$\begin{aligned}
\dim {}^X E_{q,q}^{1+} &= h_q + \binom{n}{q} \sum_{p=0}^q (-1)^{p+q} \tilde{b}_p = \\
&= h_q - (-1)^q \binom{n}{q} + \binom{n}{q} \sum_{p=0}^q (-1)^{p+q} b_p = \\
&= h_q - (-1)^q \binom{n}{q} + \binom{n}{q} \sum_{p=n-1-q}^{n-1} (-1)^{n-1-p+q} b_p = \\
&= h_{n-q} + (-1)^q \binom{n}{q} \left[-(-1)^n + (-1)^n \chi + \sum_{p=n-1-q}^{n-1} (-1)^{n-1-p} b_p \right] = \\
&= h_{n-q} + (-1)^q \binom{n}{q} \left[-(-1)^n + \sum_{p=0}^{n-q-2} (-1)^{p+n} b_p \right].
\end{aligned}$$

The final expression in brackets coincides with $\sum_{p=-1}^{n-q-2} (-1)^{p+n} \tilde{b}_p(S)$ whenever the summation is taken over nonempty set, i.e. for $q < n-1$. Thus $\dim {}^X E_{q,q}^{1+} = h'_{n-q}$ for $q < n-1$. In the case $q = n-1$ we get $\dim {}^X E_{n-1,n-1}^{1+} = h_1 + \binom{n}{n-1} = h'_1 + n$. This proves part (1) of Theorem 4.

Part (2) follows easily. Indeed, for $q = n$:

$$\dim {}^X E_{n,n}^2 = \dim {}^X E_{n,n}^1 = \binom{n}{n} \dim H_n(Q, \partial Q) = 1 = h'_0$$

For $q = n-1$:

$$\dim {}^X E_{n-1,n-1}^2 = \dim {}^X E_{n-1,n-1}^{1+} - \binom{n}{n-1} \dim \operatorname{Im} \delta_n = h'_1.$$

If $q \leq n-2$, then ${}^X E_{q,q}^2 = {}^X E_{q,q}^{1+}$, and the statement follows from part (1).

7.5. Cone case. If $Q = P(S) \cong \operatorname{Cone} |S|$, then the map $\delta_i: H_i(Q, \partial Q) \rightarrow \tilde{H}_{i-1}(\partial Q)$ is an isomorphism. Thus for $q \leq n-1$ statement 5.3 implies

$$\dim {}^X E_{q,q}^\infty = \dim {}^X E_{q,q}^{1+} - \binom{n}{q} \dim H_{q+1}(Q, \partial Q) = \dim {}^X E_{q,q}^{1+} - \binom{n}{q} \tilde{b}_q(S).$$

By Theorem 3 this expression is equal to

$$h_q(S) + \binom{n}{q} \left[\sum_{p=0}^q (-1)^{p+q} \tilde{b}_p(S) \right] - \binom{n}{q} \tilde{b}_q(S) = h_q(S) + \binom{n}{q} \sum_{p=0}^{q-1} (-1)^{p+q} \tilde{b}_p(S) = h''_q(S).$$

The case $q = n$ follows from (7.4). Indeed, the term ${}^XE_{n,n}^{1+}$ survives, thus:

$$\dim {}^XE_{n,n}^\infty = \binom{n}{n} \dim H_n(Q, \partial Q) = b_{n-1}(S) = h'_n(S) = h''_n(S).$$

This proves Theorem 5.

8. Geometry of equivariant cycles

8.1. Orientations. In this section we restrict to the case when Q is a nice manifold with corners, $X = (Q \times T^n)/\sim$ is a manifold with locally standard torus action, λ — a characteristic map over \mathbb{Z} defined on the poset $S = S_Q$. As before, suppose that all proper faces of Q are acyclic and orientable and Q itself is orientable. The subset X_I , $I \neq \emptyset$ is a submanifold of X , preserved by the torus action; X_I is called a face manifold, $\text{codim } X_I = 2|I|$. Submanifolds $X_{\{i\}}$, corresponding to vertices $i \in \text{Vert}(S)$ are called characteristic submanifolds, $\text{codim } X_{\{i\}} = 2$.

Fix arbitrary orientations of the orbit space Q and the torus T^n . This defines the orientation of $Y = Q \times T^n$ and $X = Y/\sim$. Also choose an omniorientation, i.e. orientations of all characteristic submanifolds $X_{\{i\}}$. The choice of omniorientation defines characteristic values $\omega_i \in H_1(T^n; \mathbb{Z})$ without ambiguity of sign (recall that previously they were defined only up to units of \mathbb{k}). To perform calculations with the spectral sequences XE and YE we also need to orient faces of Q .

LEMMA 8.1 (Convention). *The orientation of each simplex of S (i.e. the sign convention on S) defines the orientation of each face $F_I \subset Q$.*

PROOF. Suppose that $I \in S$ is oriented. Let i_1, \dots, i_{n-q} be the vertices of I , listed in a positive order (this is where the orientation of I comes in play). The corresponding face F_I lies in the intersecion of facets $F_{i_1}, \dots, F_{i_{n-q}}$. The normal bundles ν_i to facets F_i have natural orientations, in which inward normal vectors are positive. Orient F_I in such way that $T_x F_I \oplus \nu_{i_1} \oplus \dots \oplus \nu_{i_{n-q}} \cong T_x Q$ is positive. \square

Thus there are distinguished elements $[F_I] \in H_{\dim F_I}(F_I, \partial F_I)$. One checks that for $I <_1 J$ the maps

$$m_{I,J}^0: H_{\dim F_I}(F_I, \partial F_I) \rightarrow H_{\dim F_I-1}(\partial F_I) \rightarrow H_{\dim F_J}(F_J, \partial F_J)$$

(see (3.5)) send $[F_I]$ to $[J : I] \cdot [F_J]$. Thus the restriction maps $\mathcal{H}_0(I < J)$ send $[F_I]$ to $[F_J]$ by the definition of \mathcal{H}_0 .

The choice of omniorientation and orientations of $I \in S$ determines the orientation of each orbit $T^n/T^{\lambda(I)}$ by the following convention.

CONSTRUCTION 8.2. Let i_1, \dots, i_{n-q} be the vertices of I , listed in a positive order. Recall that $H_1(T^n/T^{\lambda(I)})$ is naturally identified with $H_1(T^n)/\mathcal{I}(I)^{(1)}$. The basis $[\gamma_1], \dots, [\gamma_q] \in H_1(T^n/T^{\lambda(I)})$, $[\gamma_l] = \gamma_l + \mathcal{I}(I)^{(1)}$ is defined to be positive if the basis $(\omega_{i_1}, \dots, \omega_{i_{n-q}}, \gamma_1, \dots, \gamma_q)$ is positive in $H_1(T^n)$. The orientation of $T^n/T^{\lambda(I)}$ determines a distinguished “volume form” $\Omega_I = \bigwedge_l [\gamma_l] \in H_q(T^n/T^{\lambda(I)}; \mathbb{Z})$.

The omniorientation and the orientation of S also determine the orientation of each manifold X_I in a similar way. All orientations are compatible: $[X_I] = [F_I] \otimes \Omega_I$.

8.2. Arithmetics of torus quotients. Let us fix a positive basis e_1, \dots, e_n of the lattice $H_1(T^n; \mathbb{Z})$. Let $(\lambda_{i,1}, \dots, \lambda_{i,n})$ be the coordinates of ω_i in this basis for each $i \in \text{Vert}(S)$. The following technical lemma will be used in subsequent computations.

LEMMA 8.3. *Let $I \in S$, $I \neq \emptyset$ be a simplex with vertices $\{i_1, \dots, i_{n-q}\}$ listed in a positive order. Let $A = \{j_1 < \dots < j_q\} \subset [n]$ be a subset of indices and $e_A = e_{j_1} \wedge \dots \wedge e_{j_q}$ the corresponding element of $H_q(T^n)$. Consider the map $\rho: T^n \rightarrow T^n/T^{\lambda(I)}$. Then $\rho_*(e_A) = C_{A,I} \Omega_I \in H_q(T^n/T^{\lambda(I)})$ with the constant:*

$$C_{A,I} = \text{sgn}_A \det (\lambda_{i,j})_{\substack{i \in \{i_1, \dots, i_{n-q}\} \\ j \in [n] \setminus A}}$$

where $\text{sgn}_A = \pm 1$ depends only on $A \subset [n]$.

PROOF. Let $(b_l) = (\omega_{i_1}, \dots, \omega_{i_{n-q}}, \gamma_1, \dots, \gamma_q)$ be a positive basis of lattice $H_1(T^n, \mathbb{Z})$. Thus $b_l = U e_l$, where the matrix U has the form

$$U = \begin{pmatrix} \lambda_{i_1,1} & \dots & \lambda_{i_{n-q},1} & * & * \\ \lambda_{i_1,2} & \dots & \lambda_{i_{n-q},2} & * & * \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \lambda_{i_1,n} & \dots & \lambda_{i_{n-q},n} & * & * \end{pmatrix}$$

We have $\det U = 1$ since both bases are positive. Consider the inverse matrix $V = U^{-1}$. Thus

$$e_A = e_{j_1} \wedge \dots \wedge e_{j_q} = \sum_{M=\{\alpha_1 < \dots < \alpha_q\} \subset [n]} \det (V_{j,\alpha})_{\substack{j \in A \\ \alpha \in M}} b_{\alpha_1} \wedge \dots \wedge b_{\alpha_q}.$$

After passing to quotient $\Lambda \rightarrow \Lambda/\mathcal{I}(I)$ all summands with $M \neq \{n-q+1, \dots, n\}$ vanish. When $M = \{n-q+1, \dots, n\}$, the element $b_{n-q+1} \wedge \dots \wedge b_n = \gamma_1 \wedge \dots \wedge \gamma_q$ goes to Ω_I . Thus

$$C_{A,I} = \det (V_{j,\alpha})_{\substack{j \in A \\ \alpha \in \{n-q+1, \dots, n\}}}.$$

Now apply Jacobi's identity which states the following (see e.g. [3, Sect.4]). Let U be an invertible $n \times n$ -matrix, $V = U^{-1}$, $M, N \subset [n]$ subsets of indices, $|M| = |N| = q$. Then

$$\det (V_{r,s})_{\substack{r \in M \\ s \in N}} = \frac{\text{sgn}_{M,N}}{\det U} \det (U_{r,s})_{\substack{r \in [n] \setminus N \\ s \in [n] \setminus M}},$$

where $\text{sgn}_{M,N} = (-1)^{\sum_{r \in [n] \setminus N} r + \sum_{s \in [n] \setminus M} s}$. In our case $N = \{n-q+1, \dots, n\}$; thus the sign depends only on $A \subset [n]$. \square

8.3. Face ring and linear system of parameters. Recall the definition of a face ring of a simplicial poset S . For $I_1, I_2 \in S$ let $I_1 \vee I_2 \subset S$ denote the set of least upper bounds, and $I_1 \cap I_2 \in S$ — the intersection of simplices (it is well-defined and unique if $I_1 \vee I_2 \neq \emptyset$).

DEFINITION 8.4. *The face ring $\mathbb{k}[S]$ is the quotient ring of $\mathbb{k}[v_I \mid I \in S]$, $\deg v_I = 2|I|$ by the relations*

$$v_{I_1} \cdot v_{I_2} = v_{I_1 \cap I_2} \cdot \sum_{J \in I_1 \vee I_2} v_J, \quad v_\emptyset = 1,$$

where the sum over an empty set is assumed to be 0.

Characteristic map λ determines the set of linear forms $\{\theta_1, \dots, \theta_n\} \subset \mathbb{k}[S]$, $\theta_j = \sum_{i \in \text{Vert}(S)} \lambda_{i,j} v_i$. If $J \in S$ is a maximal simplex, $|J| = n$, then

$$(8.1) \quad \text{the matrix } (\lambda_{i,j})_{\substack{i \leq J \\ j \in [n]}} \text{ is invertible over } \mathbb{k}$$

by the $(*_\mathbb{k})$ -condition. Thus the sequence $\{\theta_1, \dots, \theta_n\} \subset \mathbb{k}[S]$ is a linear system of parameters in $\mathbb{k}[S]$ (see, e.g., [4, lemma 3.5.8]). It generates an ideal $(\theta_1, \dots, \theta_n) \subset \mathbb{k}[S]$ which we denote by Θ .

The face ring $\mathbb{k}[S]$ is an algebra with straightening law (see, e.g. [4, §.3.5]). Additively it is freely generated by the elements

$$P_\sigma = v_{I_1} \cdot v_{I_2} \cdot \dots \cdot v_{I_t}, \quad \sigma = (I_1 \leq I_2 \leq \dots \leq I_t).$$

LEMMA 8.5. *The elements $[v_I] = v_I + \Theta$ additively generate $\mathbb{k}[S]/\Theta$.*

PROOF. Consider an element P_σ with $|\sigma| \geq 2$. Using relations in the face ring, we express $P_\sigma = v_{I_1} \cdot \dots \cdot v_{I_t}$ as $v_i \cdot v_{I_1 \setminus i} \cdot \dots \cdot v_{I_t}$, for some vertex $i \leq I_1$. The element v_i can be expressed as $\sum_{i' \leq I_t} a_{i'} v_{i'}$ modulo Θ according to (8.1) (we can exclude all v_i corresponding to the vertices of some maximal simplex $J \supseteq I_t$). Thus $v_i v_{I_t}$ is expressed as a combination of $v_{I'_t}$ for $I'_t >_1 I_t$. Therefore, up to ideal Θ , the element P_σ is expressed as a linear combination of elements $P_{\sigma'}$ which have either smaller length t (in case $|I_1| = 1$) or smaller I_1 (in case $|I_1| > 1$). By iterating this descending process, the element $P_\sigma + \Theta \in \mathbb{k}[S]/\Theta$ is expressed as a linear combination of $[v_I]$. \square

Note that the proof works for $\mathbb{k} = \mathbb{Z}$ as well.

8.4. Linear relations on equivariant (co)cycles. Let $H_T^*(X)$ be a T^n -equivariant cohomology ring of X . Any proper face of Q is acyclic, thus has a vertex. Therefore, there is the injective homomorphism

$$\mathbb{k}[S] \hookrightarrow H_T^*(X),$$

which sends v_I to the cohomology class, equivariant Poincare dual to $[X_I]$ (see [9, Lemma 6.4]). The inclusion of a fiber in the Borel construction, $X \rightarrow X \times_T ET^n$,

induces the map $H_T^*(X) \rightarrow H^*(X)$. The subspace V of $H_*(X)$, Poincare dual to the image of

$$(8.2) \quad g: \mathbb{k}[S] \hookrightarrow H_T^*(X) \rightarrow H^*(X)$$

is generated by the elements $[X_I]$, thus coincides with the ∞ -border: $V = \bigoplus_q {}^X E_{q,q}^\infty \subset H_*(X)$. Now let us describe explicitly the linear relations on $[X_I]$ in $H_*(X)$. Note that the elements $[X_I] = [F_I] \otimes \Omega_I$ can also be considered as the free generators of the \mathbb{k} -module

$$\bigoplus_q {}^X E_{q,q}^1 = \bigoplus_q \bigoplus_{|I|=n-q} H_q(F_I, \partial F_I) \otimes H_q(T^n/T^{\lambda(I)}).$$

The free \mathbb{k} -module on generators $[X_I]$ is denoted by $\langle [X_I] \rangle$.

PROPOSITION 8.6. *Let $C_{A,J}$ be the constants defined in lemma 8.3. There are only two types of linear relations on classes $[X_I]$ in $H_*(X)$:*

- (1) *For each $J \in S$, $|J| = n - q - 1$, and $A \subset [n]$, $|A| = q$ there is a relation*

$$R_{J,A} = \sum_{I >_1 J} [I : J] C_{A,I} [X_I] = 0;$$

- (2) *Let β be a homology class from $\text{Im}(\delta_{q+1}: H_{q+1}(Q, \partial Q) \rightarrow H_q(\partial Q)) \subseteq {}^{\partial Q} E_{q,0}^\infty$ for $q \leq n - 2$, and let $\sum_{|I|=n-q} B_I [F_I] \in {}^{\partial Q} E_{q,0}^1$ be a chain representing β . Then*

$$R'_{\beta,A} = \sum_{|I|=n-q} B_I C_{A,I} [X_I] = 0.$$

PROOF. This follows from the structure of the map $f_*: {}^Q E_{*,*}^* \times H_*(T^n) \rightarrow {}^X E_{*,*}^*$, lemma 8.3 and Theorem 1. Relations on $[X_I]$ appear as the images of the differentials hitting ${}^X E_{q,q}^r$, $r \geq 1$. Relations of the first type, $R_{J,A}$, are the images of $d_X^1: {}^X E_{q+1,q}^1 \rightarrow {}^X E_{q,q}^1$. In particular, $\bigoplus_q {}^X E_{q,q}^2$ is identified with $\langle [X_I] \rangle / \langle R_{J,A} \rangle$. Relations of the second type are the images of higher differentials d_X^r , $r \geq 2$. \square

Now we check that relations of the first type are exactly the relations in the quotient ring $\mathbb{k}[S]/\Theta$.

PROPOSITION 8.7. *Let $\varphi: \langle [X_I] \rangle \rightarrow \mathbb{k}[S]$ be the degree reversing linear map, which sends $[X_I]$ to v_I . Then φ descends to the isomorphism*

$$\tilde{\varphi}: \langle [X_I] \rangle / \langle R_{J,A} \rangle \rightarrow \mathbb{k}[S]/\Theta.$$

PROOF. (1) First we prove that $\tilde{\varphi}$ is well defined. The image of $R_{J,A}$ is the element

$$\varphi(R_{J,A}) = \sum_{I >_1 J} [I : J] C_{A,I} v_I \in \mathbb{k}[S].$$

Let us show that $\varphi(R_{J,A}) \in \Theta$. Let $s = |J|$, and consequently, $|I| = s + 1$, $|A| = n - s - 1$. Let $[n] \setminus A = \{\alpha_1 < \dots < \alpha_{s+1}\}$ and let $\{j_1, \dots, j_s\}$ be the vertices of J listed in a positive order. Consider $s \times (s + 1)$ matrix:

$$D = \begin{pmatrix} \lambda_{j_1, \alpha_1} & \dots & \lambda_{j_1, \alpha_{s+1}} \\ \vdots & \ddots & \vdots \\ \lambda_{j_s, \alpha_1} & \dots & \lambda_{j_s, \alpha_{s+1}} \end{pmatrix}$$

Denote by D_l the square submatrix obtained from D by deleting i -th column and let $a_l = (-1)^{l+1} \det D_l$. We claim that

$$\varphi(R_{J,A}) = \pm v_J \cdot (a_1 \theta_{\alpha_1} + \dots + a_{s+1} \theta_{\alpha_{s+1}})$$

Indeed, after expanding each θ_l as $\sum_{i \in \text{Vert}(S)} \lambda_{i,l} v_i$, all elements of the form $v_J v_i$ with $i < J$ cancel; others give $[I : J] C_{A,I} v_I$ for $I >_1 J$ according to lemma 8.3 and cofactor expansions of determinants (the incidence sign arise from shuffling columns). Thus $\tilde{\varphi}$ is well defined.

(2) $\tilde{\varphi}$ is surjective by lemma 8.5.

(3) The dimensions of both spaces are equal. Indeed, $\dim \langle [X_I] \mid |I| = n - q \rangle / \langle R_{J,A} \rangle = \dim {}^X E_{q,q}^2 = h'_{n-q}(S)$ by Theorem 4. But $\dim(\mathbb{k}[S]/\Theta)^{(n-q)} = h'_{n-q}(S)$ by Schenzel's theorem [15], [16, Ch.II, §8.2], (or [13, Prop.6.3] for simplicial posets) since S is Buchsbaum.

(4) If \mathbb{k} is a field, then we are done. This implies the case $\mathbb{k} = \mathbb{Z}$ as well. \square

In particular, this proposition describes the additive structure of $\mathbb{k}[S]/\Theta$ in terms of the natural additive generators v_I . Poincare duality in X yields

COROLLARY 8.8. *The map $g: \mathbb{k}[S] \rightarrow H^*(X)$ factors through $\mathbb{k}[S]/\Theta$ and the kernel of $\tilde{g}: \mathbb{k}[S]/\Theta \rightarrow H^*(X)$ is additively generated by the elements*

$$L'_{\beta,A} = \sum_{|I|=n-q} B_I C_{A,I} v_I$$

where $q \leq n-2$, $\beta \in \text{Im}(\delta_{q+1}: H_{q+1}(Q, \partial Q) \rightarrow H_q(\partial Q))$, $\sum_{|I|=n-q} B_I [F_I]$ is a cellular chain representing β , and $A \subset [n]$, $|A| = q$.

REMARK 8.9. The ideal $\Theta \subset \mathbb{k}[S]$ coincides with the image of the natural map $H^{>0}(BT^n) \rightarrow H_T^*(X)$. So the fact that Θ vanishes in $H^*(X)$ is not surprising. The interesting thing is that Θ vanishes by geometrical reasons already in the second term of the spectral sequence, while other relations in $H^*(X)$ are the consequences of higher differentials.

REMARK 8.10. From the spectral sequence follows that the element $L'_{\beta,A} \in \mathbb{k}[S]/\Theta$ does not depend on the cellular chain, representing β . All such chains produce the same element in $\bigoplus_q {}^X E_{q,q}^2 = \langle [X_I] \rangle / \langle R_{J,A} \rangle \cong \mathbb{k}[S]/\Theta$. Theorem 2 also implies that the relations $\{L'_{\beta,A}\}$ are linearly independent in $\mathbb{k}[S]/\Theta$ when β runs over some basis of $\text{Im } \delta_{q+1}$ and A runs over all subsets of $[n]$ of cardinality q .

9. Examples and calculations

9.1. Quasitoric manifolds. Let Q be n -dimensional simple polytope. Then $S = S_Q = \partial Q^*$ is the boundary of the polar dual polytope. In this case $Q \cong \text{Cone } |S|$. Given a characteristic map $\lambda: \text{Vert}(K) \rightarrow \mathcal{T}_n$ we construct a space $X = (Q \times T^n)/\sim$ which is a model of quasitoric manifold [7]. Poset S is a sphere thus $h''_i(S) = h'_i(S) = h_i(S)$. Since $\delta_n: H_n(Q, \partial Q) \rightarrow H_{n-1}(\partial Q)$ is an isomorphism, Theorem 2 implies ${}^X E_{p,q}^2 = 0$ for $p \neq q$. By Theorems 3 and 5, $\dim {}^X E_{q,q}^2 = h_q(S) = h_{n-q}(S)$. Spectral sequence ${}^X E_{*,*}^*$ collapses at its second term, thus $\dim H_{2q}(X) = h_q(S) = h_{n-q}(S)$ for $0 \leq q \leq n$ and $\dim H_{2q+1}(X) = 0$ which is well known. For bigraded Betti numbers proposition 5.6 implies $H_{i,j}(X) = 0$ if $i \neq j$, and $\dim H_{i,i}(X) = h_i(S)$.

9.2. Homology polytopes. Let Q be a manifold with corners such that all its proper faces as well as Q itself are acyclic. Such objects were called homology polytopes in [9]. In this case everything stated in the previous paragraph remains valid, thus $\dim H_{2q}(X) = h_q(S) = h_{n-q}(S)$ for $0 \leq q \leq n$, and $\dim H_{2q+1}(X) = 0$ (see [9]).

9.3. Origami toric manifolds. Origami toric manifolds appeared in differential geometry as generalizations of symplectic toric manifolds (see [5],[10]). The original definition contains a lot of subtle geometrical details and in most part is irrelevant to this paper. Here we prefer to work with the ad hoc model, which captures most essential topological properties of origami manifolds.

DEFINITION 9.1. *Topological toric origami manifold X^{2n} is a manifold with locally standard action $T^n \curvearrowright X$ such that all faces of the orbit space including X/T itself are either contractible or homotopy equivalent to wedges of circles.*

As before consider the canonical model. Let Q^n be a nice manifold with corners in which every face is contractible or homotopy equivalent to a wedge of b_1 circles. Every principal T^n -bundle Y over Q is trivial (because $H^2(Q) = 0$), thus $Y = Q \times T^n$. Consider the manifold $X = Y/\sim$ associated to some characteristic map over \mathbb{Z} . Then X is a topological origami toric manifold.

To apply the theory developed in this paper we also assume that all proper faces of Q are acyclic (in origami case this implies contractible) and Q itself is orientable. Thus, in particular, Q is a Buchsbaum manifold. First, describe the exact sequence of the pair $(Q, \partial Q)$. By Poincaré–Lefschetz duality:

$$H_q(Q, \partial Q) \cong H^{n-q}(Q) \cong \begin{cases} \mathbb{k}, & \text{if } q = n; \\ H^1(\bigvee_{b_1} S^1) \cong \mathbb{k}^{b_1}, & \text{if } q = n - 1; \\ 0, & \text{otherwise.} \end{cases}$$

In the following let m denote the number of vertices of S (the number of facets of Q). Thus $h'_1(S) = h_1(S) = m - n$. Consider separately three cases:

(1) $n = 2$. In this case Q is an orientable 2-dimensional surface of genus 0 with $b_1 + 1$ boundary components. Thus ∂Q is a disjoint union of $b_1 + 1$ circles and long exact sequence in homology has the form:

$$\begin{array}{ccccccc}
 \begin{array}{c} 0 \\ \parallel \\ H_2(Q) \end{array} & \longrightarrow & \begin{array}{c} \mathbb{k} \\ \parallel \\ H_2(Q, \partial Q) \end{array} & \xrightarrow{\delta_2} & \begin{array}{c} \mathbb{k}^{b_1+1} \\ \parallel \\ H_1(\partial Q) \end{array} & \longrightarrow & \begin{array}{c} \mathbb{k}^{b_1} \\ \parallel \\ H_1(Q) \end{array} \xrightarrow{0} \\
 & & & & & & \xrightarrow{0} \begin{array}{c} H_1(Q, \partial Q) \\ \parallel \\ \mathbb{k}^{b_1} \end{array} \xrightarrow{\delta_1} \begin{array}{c} H_0(\partial Q) \\ \parallel \\ \mathbb{k}^{b_1+1} \end{array} \longrightarrow \begin{array}{c} H_0(Q) \\ \parallel \\ \mathbb{k} \end{array} \longrightarrow \begin{array}{c} H_0(Q, \partial Q) \\ \parallel \\ 0 \end{array}
 \end{array}$$

The second term ${}^X E_{*,*}^2$ of spectral sequence for X is given by Theorem 2. It is shown on a figure below (only ranks are written to save space).

2			1
1		$m - 2$	b_1
0	$b_1 + 1$	b_1	$2b_1$
-1			b_1
	0	1	2

The only nontrivial higher differential is $d^2: {}^X E_{2,-1}^2 \rightarrow {}^X E_{0,0}^2$; it coincides with the composition of $\delta_1 \otimes \text{id}_{H_0(T^2)}$ and injective map $f_*^2: H_0(P) \otimes H_0(T^2) \rightarrow {}^X E_{0,0}^2$. Thus d^2 is injective, and $\dim {}^X E_{2,2}^\infty = \dim {}^X E_{0,0}^\infty = 1$; $\dim {}^X E_{2,1}^\infty = \dim {}^X E_{1,0}^\infty = b_1$; $\dim {}^X E_{1,1}^\infty = m - 2$; $\dim {}^X E_{2,0}^\infty = 2b_1$. Finally,

$$\dim H_i(X) = \begin{cases} 1, & \text{if } i = 0, 4; \\ b_1, & \text{if } i = 1, 3; \\ m - 2 + 2b_1, & \text{if } i = 2. \end{cases}$$

This coincides with the result of computations in [14], concerning the same object. This result can be obtained simply by proposition 5.6: $\dim H_{0,0}(X) = \dim H_{2,2}(X) = 1$, $\dim H_{1,0}(X) = \dim H_{1,2}(X) = b_1$, $\dim H_{2,2}(X) = m - 2 + 2b_1$.

(2) $n = 3$. In this case the exact sequence of $(Q, \partial Q)$ splits in three essential parts:

$$\begin{array}{ccccccc}
 H_3(Q) & \longrightarrow & H_3(Q, \partial Q) & \xrightarrow{\delta_3} & H_2(\partial Q) & \longrightarrow & H_2(Q) \\
 \parallel & & \parallel & & & & \parallel \\
 0 & & \mathbb{k} & & & & 0 \\
 \\
 H_2(Q) & \longrightarrow & H_2(Q, \partial Q) & \xrightarrow{\delta_2} & H_1(\partial Q) & \longrightarrow & H_1(Q) \longrightarrow H_1(Q, \partial Q) \\
 \parallel & & \parallel & & \parallel & & \parallel \\
 0 & & \mathbb{k}^{b_1} & & \mathbb{k}^{b_1} & & 0 \\
 \\
 H_1(Q, \partial Q) & \xrightarrow{\delta_1} & H_0(\partial Q) & \longrightarrow & H_0(Q) & \longrightarrow & H_0(Q, \partial Q) \\
 \parallel & & & & \parallel & & \parallel \\
 0 & & & & \mathbb{k} & & 0
 \end{array}$$

By Theorems 2, 4, ${}^X E_{p,q}^2$ has the form

3				1
2			h'_1	b_1
1		h'_2	0	$3b_1$
0	h'_3	$2b_1$	0	$3b_1$
-1				b_1
	0	1	2	3

There are two nontrivial higher differentials: $d^2: {}^X E_{3,0}^2 \rightarrow {}^X E_{1,1}^2$ and $d^2: {}^X E_{3,-1}^2 \rightarrow {}^X E_{1,0}^2$; both are injective. Thus $\dim {}^X E_{3,3}^\infty = \dim {}^X E_{0,0}^\infty = 1$; $\dim {}^X E_{3,1}^\infty = \dim {}^X E_{1,0}^\infty = b_1$; $\dim {}^X E_{2,2}^\infty = h'_1$; $\dim {}^X E_{3,1}^\infty = 3b_1$; $\dim {}^X E_{1,1}^\infty = h'_2 - 3b_1$. Therefore,

$$\dim H_i(X) = \begin{cases} 1, & \text{if } i = 0, 6; \\ b_1, & \text{if } i = 1, 5; \\ h'_1 + 3b_1, & \text{if } i = 4; \\ h'_2 - 3b_1, & \text{if } i = 2; \\ 0, & \text{if } i = 3. \end{cases}$$

(3) $n \geq 4$. In this case lacunas in the exact sequence for $(Q, \partial Q)$ imply that $\delta_i: H_i(Q, \partial Q) \rightarrow H_{i-1}(\partial Q)$ is an isomorphism for $i = n-1, n$, and is trivial otherwise.

We have

$$H_i(\partial Q) \cong \begin{cases} H_n(Q, \partial Q) \cong \mathbb{k}, & \text{if } i = n - 1; \\ H_{n-1}(Q, \partial Q) \cong \mathbb{k}^{b_1}, & \text{if } i = n - 2; \\ H_1(Q) \cong \mathbb{k}^{b_1} & \text{if } i = 1; \\ H_0(Q) \cong \mathbb{k}, & \text{if } i = 0; \\ 0, & \text{o.w.} \end{cases}$$

By Theorems 2 and 4, ${}^X E_{p,q}^2$ has the form

						1
					h'_1	$\binom{n}{n}b_1$
				$\textcircled{h'_2}$	0	$\binom{n}{n-1}b_1$
			h'_3	$\binom{n}{n-3}b_1$	0	$\binom{n}{n-2}b_1$
		\ddots		\vdots		$\binom{n}{n-3}b_1$
	h'_{n-1}	0	0	$\binom{n}{1}b_1$	0	\vdots
	h'_n	$\binom{n}{0}b_1$	0	0	$\binom{n}{0}b_1$	$\binom{n}{1}b_1$
-1						$\binom{n}{0}b_1$
0						

Thus we get: $\dim {}^X E_{q,q}^\infty = h'_{n-q}$, if $q \neq n - 2$; $\dim {}^X E_{n-2,n-2}^\infty = h'_2 - \binom{n}{2}b_1$ if $q = n - 2$; $\dim {}^X E_{n,n-1}^\infty = \dim {}^X E_{1,0}^\infty = b_1$; $\dim {}^X E_{n,n-2}^\infty = nb_1$. Finally, by proposition 5.6, $\dim H_{1,0}(X) = \dim H_{n-1,n}(X) = b_1$, $\dim H_{n-1,n-1}(X) = h'_1 + nb_1$, $\dim H_{n-2,n-2}(X) = h'_2 - \binom{n}{2}b_1$, and $\dim H_{i,i}(X) = h'_{n-i}$ for $i \neq n - 1, n - 2$.

The differential hitting the marked position produces additional relations (of the second type) on the cycles $[X_I] \in H^{2n-4}(X)$. These relations are described explicitly by proposition 8.6. Dually, this consideration shows that the map $\mathbb{k}[S]/\Theta \rightarrow H^*(X)$ has a nontrivial kernel only in degree 4. The generators of this kernel are described by corollary 8.8.

10. Concluding remarks

Several questions concerning the subject of this paper are yet to be answered.

- (1) Of course, the main question which remains open is the structure of multiplication in the cohomology ring $H^*(X)$. The border module $\bigoplus_q {}^X E_{q,q}^\infty \subset H_*(X)$ represents an essential part of homology; the structure of multiplication on the corresponding subspace in cohomology can be extracted from the ring homomorphism $\mathbb{k}[S]/\Theta \rightarrow H^*(X)$. Still there are cocycles which do not come from $\mathbb{k}[S]$ and their products should be described separately. Proposition 5.6 suggests, that some products can be described via the multiplication in $H^*(Q \times T^n) \cong H^*(Q) \otimes H^*(T^n)$. This requires further investigation.
- (2) It is not clear yet, if there is a torsion in the border module $\bigoplus_q {}^X E_{q,q}^\infty$ in case $\mathbb{k} = \mathbb{Z}$. Theorems 3,4,5 describe only the rank of the free part of this group, but the structure (and existence) of torsion remains open. Note that the homology of X itself can have a torsion. Indeed, the groups $H_*(Q)$, $H_*(Q, \partial Q)$ can contain arbitrary torsion, and these groups appear in the description of $H_*(X)$ by proposition 5.6.
- (3) Corollary 8.8 describes the kernel of the map $\mathbb{k}[S]/\Theta \rightarrow H^*(X)$. It seems that the elements of this kernel lie in a socle of $\mathbb{k}[S]/\Theta$, i.e. in a submodule $\{x \in \mathbb{k}[S]/\Theta \mid (\mathbb{k}[S]/\Theta)^+ x = 0\}$. The existence of such elements is guaranteed in general by the Novik–Swartz theorem [13]. If the relations $L'_{\beta,A}$ do not lie in a socle, their existence would give refined inequalities on h -numbers of Buchsbaum posets.
- (4) Theorem 6 establish certain connection between the sheaf of ideals generated by linear elements and the cosheaf of ideals generated by exterior products. This connection should be clarified and investigated further. In particular, statement 6.6 can probably lead to the description of homology for the analogues of moment-angle complexes, i.e. the spaces of the form $X = Y/\sim$, where Y is an arbitrary principal $T^{\mathbb{N}}$ -bundle over Q .
- (5) There is a hope, that the argument of section 6 involving two spectral sequences for a sheaf resolution can be generalized to non-Buchsbaum case.
- (6) The real case, when $T^{\mathbb{N}}$ is replaced by $\mathbb{Z}_2^{\mathbb{N}}$, can, probably, fit in the same framework.

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OSAKA CITY UNIVERSITY

E-mail address: ayzenberga@gmail.com